Actuarial Mathematics (MA310)

10.10am Mondays AC202
12.10pm Fridays C219

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A good performance in this course will give you an exemption from the Actuarial Exam

CT1 - Mathematics of Finance

Outline

- Fundamentals
- Project appraisal
- Investment
- Simple compound interest problems
- Arbitrage and forward contracts
- Term structure of interest rates

Reading

- "Core Reading 2007 - CT1 Financial Mathematics", The Official Actuarial Bookshop
- "An introduction to the mathematics of finance", John J. McCutcheon and William F. Scott, London: Heinemann, 1986. ISBN: 0 434 91228 x

Calculator

Get yourself a good calculator which is NOT capable of storing text.

Section A: Fundamentals

Interest rates

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- $v = \frac{1}{1+i}$ = amount that grows to 1 at the end of $\overline{1}|$ at ROI i.

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- $d^{(m)}$ =Nominal rate of discount convertible $m$thly. Effective ROD for each $1/m$ period $= d^{(m)}/m$. Thus, by definition,

$$(1 - \frac{d^{(m)}}{m})^m = 1 - d$$

$$i,\ i^{(m)},\ \delta,\ v,\ d,\ d^{(m)}$$

Different forms of the same thing.
Use the form most appropriate to the task in hand.

Example.
If the nominal rate of interest convertible every month is 8% p.a., calculate the nominal rate of interest p.a. convertible every quater and the effective rate of discount p.a. . [5 minutes]

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and
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- $$d = 1 - v = 1 - \frac{1}{1+i} = .076639$$

Cashflow models

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- Cashflow model: model that describes the timing and amount of the CF

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	Cashflows	
	5 per week	Winnings
Syndicate's viewpoint		
Cashflow sign	(-)	(+)
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Syndicate's viewpoint		
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Lotto's viewpoint		
Cashflow sign	(+)	(-)
Timing	unknown	unkown
Amount	unknown	unkown

- It can be useful to plot cashflows on a timeline.

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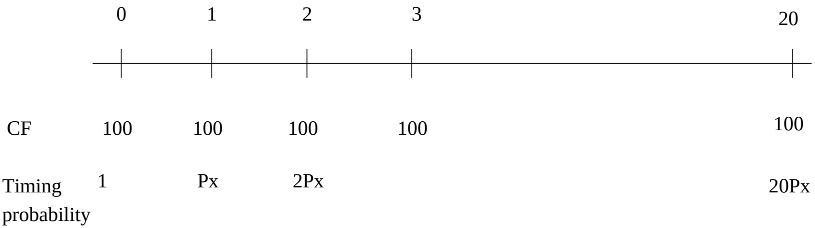
- Where there are uncertainties it is possible to assign probabilities to both the amount and the existence of a cashflow

Example: Life assurance policy (Insurance company view). Whole of life non profit. Premium of £100 pa.

Cashflow	Sign	Timing	Amount
Premium	(+)	unknown	known
Sum assured	(-)	unkown	known
Expenses	(-)	known	unkown



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- $\overline{n}|$ @d% pa simple discount: $(1-nd)C \to C$
- $\overline{n}|$ @d% pa compound discount: $(1-d)^n C \to C$

Example

A company wishes to invest £10,000 in 89-day bills from the British government. The bills are currently issued at a simple rate of discount of 7% per annum. Calculate the nominal value of the bills that can be purchased. [5 minutes]

N = nominal value

$$(1 - 0.07\frac{89}{365})N = 10,000$$

N = £10,173.65

Present values

PV now of an amount X at time $n = X/(1+i)^n = v^n X$

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A loan of £2400 is to be repaid in 20 equal installments. The rate of interest for the transaction is 10% per annum. Find the amount of each annual repayment assuming that payments are made (a) in arrear, (b) in advance.

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$$X = 281.90$$

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$$Y = 256.28$$

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Increasing & decreasing annuities

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- $(Ia)_{\overline{n}|} = \text{PV of } (1@t=1) \ + \ (2@t=2) \ + \cdots + \ (n@t=n)$
  $(Ia)_{\overline{n}|} = v + 2v^2 + 3v^3 + \cdots + nv^n$

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- $$(I\ddot{a})_{\overline{n}|} = 1 + 2v + 3v^2 + \cdots + nv^{n-1} = (1+i)(Ia)_{\overline{n}|}$$

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- $$(\overline{Ia})_{\overline{n}|} = \int_0^n tv^t\,dt = \frac{\overline{a}_{\overline{n}|} - nv^n}{\delta}$$
- $$(I\overline{a})_{\overline{n}|} = \sum_{r=1}^{n}(\int_{r-1}^{r} rv^t\,dt) = \frac{\overline{a}_{\overline{n}|}}{a_{\overline{n}|}}(Ia)_{\overline{n}|}$$

Increasing & decreasing annuities contd.

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- $$(Da)_{\overline{n}|} = nv + (n-1)v^2 + (n-2)v^3 + \cdots + v^n = \frac{n - a_{\overline{n}|}}{i}$$

Increasing & decreasing annuities contd.

- $$(Da)_{\overline{n}|} = nv + (n-1)v^2 + (n-2)v^3 + \cdots + v^n = \frac{n - a_{\overline{n}|}}{i}$$
- and so on ...

Example

Show that

$$(\overline{Ia})_{\overline{n}|} = \frac{\overline{a}_{\overline{n}|} - nv^n}{\delta} \ .$$

[5 minutes]

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[5 minutes]

- $$\int_0^n t(v^n\,dt) = (t(-\frac{1}{\delta}e^{-\delta})_0^n - \int_0^n -\frac{1}{\delta}e^{-\delta}\,dt$$
$$= -\frac{nv^n}{\delta} + \frac{1}{\delta}\int_0^n v^t\,dt$$

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- $$= -\frac{nv^n}{\delta} + \frac{1}{\delta}\overline{a}_{\overline{n}|}$$