Actuarial Mathematics (MA310)

Graham Ellis
http://hamilton.nuigalway.ie

Arbitrage & Forward Contracts

Arbitrage

An Arbitrage exists if either

(a) An investor can make a deal that would give an immediate profit with no risk of future loss.

(b) Or an investor can make a deal that has zero initial cost, no risk of future loss, and a non-zero probability of a future profit.

Example (a)

There are two securities A and B. At time $t = 0$ the securities cost P_0^A and P_0^B respectively. The term of the securities is 1 year. At $t = 1$ either the market goes up and the securities pay $P_1^A(u)$, $P_1^B(u)$ or the market goes down with payments $P_1^A(d)$, $P_1^B(d)$. Investors buy a security by paying the time 0 price and receiving time 1 income. Investors sell a security by receiving the time 0 price and paying time 1 outgo. Assume

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Whatever the market: profit of 1 at time 0 with no future loss.

Example (b)

As above with table

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Arbitrage opportunity: Buy 1 of A and sell 1 of B.

The investor has a possibility of making a profit and no possibility of making a loss.

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In practice arbitrage opportunities do arise but are fleeting in nature. So it is prudent to assume they don't exist.

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- $\delta =$ (known) force of interest available ona risk-free investment over the term of the forward contract.
- K is determined such that the value of the forward contract at $t = 0$ is zero (i.e. no arbitrage assumption). Profit at time $t = T$ for buyer is $S_T - K$.

Calculating the forward price for a security with NO income
The no arbitrage assumption implies the following portfolios must be equal in value:

Portfolio A: Enter into a forward contract to buy one unit of S with forward price K maturing at time $t = T$ AND invest an amount $Ke^{-\delta T}$ at time $t = 0$.

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So $Ke^{-\delta T} = S_0$ or

$$K = S_0 e^{\delta T}.$$

Example

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$e^{\delta} = (1+i) \Rightarrow e^{\delta T} = (1+i)^T$. So

$$K = 4.26e^{\frac{3}{12}\delta}$$

$$K = 42.6(1.06)^{\frac{3}{12}} = 43.23$$

Calculating the forward price for a security with fixed cash income

Assume that the security underlying the forward contract provides a fixed amount c at $t = t_1$, $0 < t_1 < T$.

Portfolio A: Forward contract to buy 1 unit of S at price K at $t = T$ AND invest an amount $Ke^{-\delta T} + ce^{-\delta t_1}$ in risk-free investment.

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At $t = T$: CFs from A $=< K > +K + ce^{\delta(T-t_1)}$ to receive 1 unit of S of value S_T.
At $t = T$: CFs from B $= ce^{\delta(T-t_1)}$ to hold 1 unit of S of value S_T.

Portfolios A & B generate identical cashflows. So the no arbitrage assumption implies

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If there is more than one income payment it can be shown that

$$K = (S_0 - I)e^{\delta T}$$

where I is the PV at $t = 0$ of the fixed income payments.

Example

A fixed interest security pays coupons of 8% per annum half-yearly in arrears and is redeemable at 110%. Two months before the next coupon is due, an investor negotiates a forward contract in which he agrees to buy £60,000 nominal of the security in 6 months time. The current price of the stock is £80.40 per £100 nominal, and the risk-free force of interest is 5% per annum. Calculate the forward price.

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$$K = (80.4)600e^{.05(\frac{6}{12})} - \frac{.08}{2}60,000e^{.05(\frac{6}{12}-\frac{2}{12})}$$

$$= 47,021$$

(Note: the 110% redemption rate is not required to calculate K.)

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$$= \left((80.4)(500) - \frac{.08}{2}50,000(e^{-0.5\frac{2}{12}} + e^{-0.5\frac{8}{12}}\right) e^{.05(\frac{10}{12})}$$

$$= 37,826$$

Forward price for a security with known dividend yield

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CF A $= < K > + Ke^{-\delta T} e^{\delta T} = 0$ to receive 1 unit of S
CF B $= 0$ and hold $e^{-DT} e^{DT} = 1$ unit of S

Portfolios A & B generate equal cash flows. So by the "no arbitrage" assumption $P_0^A = P_0^B$. Thus

$Ke^{-\delta T} = e^{-DT} S_0$

or

$$K = S_0 e^{(\delta - D)T}$$

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$$\delta = \ln(1+i) = \ln(1.046028) = 0.045$$

$$K = 673,000e^{(.045-.028)1} = 684,539$$

It can be shown that the forward price K when dividends are received at the end of each year and immediately re-invested and T is an integer, is given by

$$K = S_0 e^{\delta T}(1 + D)^{-T}$$

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- The preceding examples have been of the "static hedge" type since the hedge portfolio, which consists of the asset to be sold plus the borrowed risk-free investment, does not change over the term of the contract.
- More complicated financial instruments require the hedge portfolio to be continuously rebalanced (dynamic hedging).

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At $t = r$, $0 < r < T$, value of forward contract $=$?

Value of a Long Forward Contract (to buyer at $t = T$)

At $t = r$

Portfolio A: Buy existing long FC for V_L AND invest $K_0 e^{\delta(T-r)}$ risk-free for $T - r$ years.

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Using $K_0 = S_0 e^{\delta T}$ and $K_r = S_r e^{\delta(T-r)}$ we get

$$V_L = S_r - S_0 e^{\delta r}$$

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On 1 January 1999 an investor agrees to pay £3000 in four years time for a security. The security pays no interest and the price of the security at the time of the agreement was £2,680 . On 1 July 2000 the price of the security is £2800. Calculate the value of the forward contract on 1 July 2000.

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On 1 July 2000:

$$V_L = S_r - S_0 e^{\delta r} = 2800 - 2680 e^{1.5(.0282)} = 4.20$$