MA286: Tutorial Problems 2015-16

Tutorials: Tuesday, 6-7pm, Venue = IT202
Thursday, 6-7pm, Venue = IT204

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For those questions taken from the Schaum Outline Series book *Advanced Calculus* by M. Spiegel the question number in the book is given. The book provides worked solutions for many of these questions. I'll add problems to this sheet from time to time throughout the semester.

PROBLEMS

1 0-forms on 1-dimensional space

1. Give an interval $S = [a, b] \subset \mathbb{R}$ on which
$$\omega = |x - 4|$$
is a differential 0-form. Then give an interval $S' = [a', b'] \subset \mathbb{R}$ on which ω is not a differential 0-form.

2. Evaluate the integral
$$\int_{\partial S} 2x^2 + x$$
of the differential 0-form $\omega = 2x^2 + x$ over the boundary of the oriented interval $S = [3, -1]$.

3. Evaluate the integral
$$\int_{\partial S} x^3$$
of the differential 0-form $\omega = x^3$ over the boundary of $S = [2, 1] \cup [4, 3] \cup [-2, -1]$.

4. Is
$$\omega = \begin{cases} x \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$
a differential 0-form on the interval $S = [-1, 1]$? [See 4.4(b)]

5. Is the function
$$\omega = \begin{cases} x^2 \cos(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$
differentiable at $x = 0$?

2 1-forms on 1-dimensional space

1. Find a differential 0-form ω whose derivative $d\omega$ is the differential 1-form
$$d\omega = (x^2 + 2x)\, dx\ .$$

2. Find a differential 0-form ω whose derivative $d\omega$ is the differential 1-form
$$d\omega = (x + 2)\sin(x^2 + 4x - 6)\, dx\ .$$
[See 5.14(a)]

3. Find a differential 0-form ω whose derivative $d\omega$ is the differential 1-form
$$d\omega = \frac{6 - x}{(x - 3)(2x + 5)}\, dx\ .$$
[See 5.20]

4. Find a differential 0-form ω whose derivative $d\omega$ is the differential 1-form
$$d\omega = \frac{1}{5 + 3\cos x}\, dx\ .$$
[See 5.21]

3 Fundamental theorem of calculus

1. Evaluate the integral
$$\int_S \frac{1}{\sqrt{(x + 2)(3 - x)}}\, dx$$
of the differential 1-form $\omega = dx/\sqrt{(x + 2)(3 - x)}$ over the oriented interval $S = [-1, 1]$. [See 5.14(c)]

2. Evaluate the integral
$$\int_S \frac{1}{(x^2 - 2x + 4)^{3/2}}\, dx$$
of the differential 1-form $\omega = (x^2 - 2x + 4)^{-3/2}\, dx$ over the oriented interval $S = [2, 1]$. [See 5.15]

3. Evaluate the integral
$$\int_S \frac{1}{x(\ln x)^3}\, dx$$
of the differential 1-form $\omega = dx/x(\ln x)^3$ over the oriented interval $S = [e, e^2]$. [See 5.16]

4. Give an informal proof of Stokes' formula $\int_{\partial S} \omega = \int_S d\omega$ for $S = [a, b] \subset \mathbb{R}$ and $\omega = f(x)\colon \mathbb{R} \to \mathbb{R}$ a differentiable function.

4 0-forms on n-dimensional space

1. Let S denote the oriented line segment in the plane going from the point $A = (1, 2)$ to the point $B = (-2, 3)$. Evaluate the integral

$$\int_{\partial S} x^2 + xy + y^2$$

   of the differential 0-form $\omega = x^2 + xy + y^2$ over the boundary of S.

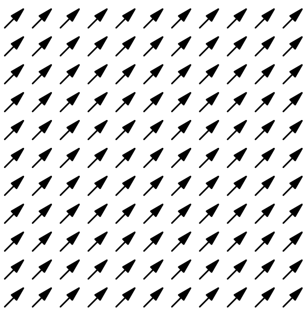
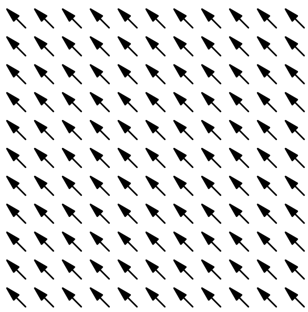
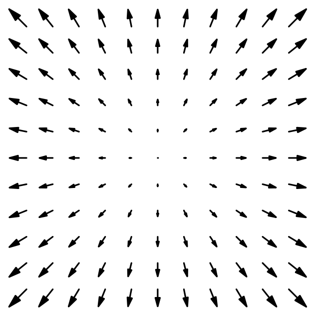
2. Let S denote the oriented line segment on the z-axis in $\mathbb{R}^3$ going from $z = 1$ to $z = 2$. Evaluate the integral

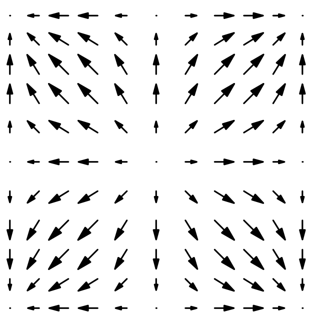
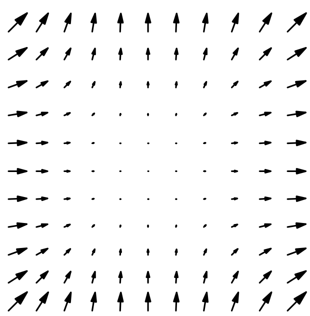
$$\int_{\partial S} z\, e^{x^2+y^2}$$

   of the differential 0-form $\omega = z\, e^{x^2+y^2}$ over the boundary of S.

5 1-forms on n-dimensional space

1. Match the five pictures of flows

(a) 
(b) 
(c) 

(d) 
(e) 

   to the five differential 1-forms: (i) $\omega = x^2\, dx + y^2\, dy$, (ii) $\omega = \sin(\pi x)\, dx + \sin(\pi y)\, dy$, (iii) $\omega = x\, dx + y\, dy$, (iv) $\omega = dx + dy$, (v) $\omega = -dx + dy$.

2. In a constant force field the displacement of a particle

   - from $(0, 0, 0)$ to $(4, 0, 0)$ needs 3 units of work;

- from $(1,-1,0)$ to $(1,1,0)$ needs 2 units of work;
- from $(0,0,0)$ to $(3,0,2)$ needs 5 units of work.

Determine the differential 1-form that describes "work".

6 Integration of constant 1-forms

1. Evaluate the integral
$$\int_S 2\,dx + 3\,dy + 5\,dz$$
of the differential 1-form $\omega = 2\,dx + 3\,dy + 5\,dz$ on the line segment S in $\mathbb{R}^3$ starting at point $P = (3, 12, 4)$ and ending at point $Q = (11, 14, -7)$.

2. If work is given by the 1-form $3\,dx + 4\,dy - dz$ find all points which can be reached from the origin $(0,0,0)$ without work. Describe the set of these points geometrically.

The first test will be based on the problems up to this point.

7 Integration of 1-forms

1. Evaluate the integral
$$\int_S (x^2 - y)\,dx + (y^2 + x)\,dy$$
of the differential 1-form $\omega = (x^2 - y)\,dx + (y^2 + x)\,dy$ where $S \subset \mathbb{R}^2$ is the segment of the parabola $x = t$, $y = t^2 + 1$ from the point $(0,1)$ to the point $(1,2)$. [See 10.1]

2. Evaluate the integral
$$\int_C \omega$$
of the 1-form
$$\omega = (3x^2 - 6yz)\,dx + (2y + 3xz)\,dy + (1 - 4xyz^2)\,dz$$
where C is the straight line from $(0,0,0)$ to $(1,1,1)$. [See 10.2(c)]

3. Evaluate the integral
$$\int_C \omega$$
of the 1-form
$$\omega = (3x^2 - 6yz)\,dx + (2y + 3xz)\,dy + (1 - 4xyz^2)\,dz$$
where C is the curve $x = t$, $y = t^2$, $z = t^3$ from $(0,0,0)$ to $(1,1,1)$. [See 10.2(c)]

4. Evaluate the integral

$$\int_{\partial S} (2xy - x^2)\, dx + (x + y^2)\, dy$$

where ∂S is the boundary of the region S bounded by the two curves $y = x^2$ and $y^2 = x$. Assume an anti-clockwise orientation on ∂S. [See 10.6]

8 Differentiation of 0-forms

1. Determine the 1-form $d\omega$ arising as the derivative of the 0-form $\omega = x^2 e^{y/x}$. [See 6.16(a)]

2. Find a 0-form ω whose derivative is

$$d\omega = (3x^2 y - 2y^2)\, dx + (x^3 - 4xy + 6y^2)\, dy.$$

[See 6.16(b)]

9 Partial derivatives

1. Suppose $U = z \sin(y/x)$ where $x = 3r^2 + 2s$, $y = 4r - 2s^3$ and $z = 2r^2 - 3^2$. Calculate $\partial u/\partial r$ and $\partial U/\partial s$. [See 6.22]

10 Fundamental Theorem of Calculus again

1. Evaluate

$$\int_S (6xy^2 - y^3)dx + (6x^2 y - 3xy^2)dy\ .$$

where S is some path from $(1, 2)$ to $(3, 4)$. Explain why the integral is independent of the choice of path from $(1, 2)$ to $(3, 4)$. [See 10.14]

2. Evaluate

$$\int_S (2xy - y^4 + 3)dx + (x^2 - 4xy^3)dy\ .$$

where S is some path from $(1, 0)$ to $(2, 1)$. Explain why the integral is independent of the choice of path from $(1, 0)$ to $(2, 1)$. [See 10.48]

3. Prove that the differential 1-form

$$\omega = (3x^2 - 6yz)\, dx + (2y + 3xz)\, dy + (1 - 4xyz^2)\, dz$$

does not arise as the derivative $\omega = d\nu$ of any 0-form ν on $S - \mathbb{R}^3$. [See Questions 2 and 3 of Section 7 above]

4. Prove Stokes' formula $\int_{\partial S} \omega = \int_S d\omega$ for $\omega = f(x, y) \colon \mathbb{R}^2 \to \mathbb{R}$ a continuously differentiable function and $S \subset \mathbb{R}^2$ an oriented curve with differentiable parametrization $x = g(t)$, $y = h(t)$.

11 Constant 2-forms

1. Evaluate the integral

$$\int_S dx \wedge dy + 3dx \wedge dz$$

of the 2-form $\omega = dx \wedge dy + 3dx \wedge dz$ over the oriented triangle S with vertices $(0, 0, 0)$, $(1, 2, 3)$, $(1, 4, 0)$ in that order.

2. Evaluate the integral

$$\int_S dy \wedge dz + dz \wedge dx + dx \wedge dy$$

of the 2-form $\omega = dy \wedge dz + dz \wedge dx + dx \wedge dy$ over the oriented triangle S with vertices $(1, 1, 1)$, $(3, 5, -1)$, $(4, 2, 1)$ in that order.

3. Evaluate the integral

$$\int_S 3dx \wedge dy$$

of the 2-form $\omega = 3dx \wedge dy$ over the region $S = \{(x, y) \in \mathbb{R}^2 \ : \ x^2 + y^2 \leq 1\}$ where S is given a clockwise rotation when viewed from the positive z-axis.

12 More integration of 2-forms

1. Evaluate the integral

$$\int_S 3\, dx \wedge dy + 4\, dy \wedge dz$$

over the region $S = \{(x, y) \in \mathbb{R}^2 \ : \ x^2 + y^2 \leq 1\}$ where S is given a clockwise rotation when viewed from the positive z-axis.

2. Let S be the region in the xy-plane bounded by $y = x^2$, $x = 2$ and $y = 1$. Let S have an anti-clockwise orientation. Evaluate

$$\int_S (x^2 + y^2 + z^2)\, dx \wedge dy.$$

[See 9.1]

3. Let S be the region in the xy-plane bounded by $y = x^2$, $x = 2$ and $y = 1$. Let S have an anti-clockwise orientation. Evaluate

$$\int_S (x^2 + y^2 + z^2)\, dy \wedge dz.$$

4. Let S be the region in the xy-plane bounded by the curves $y = x^2$, $y = \sqrt{2 - x^2}$, $x = 0$ and $x = 1$. Let S have an anti-clockwise orientation. Evaluate

$$\int_S xy\, dx \wedge dy.$$

[See 9.3(b)]

5. Find the volume of the region in $\mathbb{R}^3$ common to the intersecting cylinders $x^2 + y^2 = 4$ and $x^2 + z^2 = 4$. [See 9.4]

The second test will be based on all the problems up to this point, including those covered by the first test.

13 Differentiation of k-forms

1. Find $d\omega$ for the following forms.

   (a) $\omega = xy\, dz + yz\, dx + zx\, dy$

   (b) $\omega = x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy$

   (c) $\omega = e^{xyz}$

   (d) $\omega = (\cos x)\, dy + (\sin x)\, dz$

   (e) $\omega = (x + y)^2\, dy + (x + y)^2\, dz$

   (f) $\omega = \log x$

   (g) $\omega = x^2$

   (h) $\omega = \sin x$

(i) $\omega = x$

2. Let $\omega = F(x,y,z)$ be a 0-form and assume $F_{xy} = F_{yx}$, $F_{xz} = F_{zx}$, $F_{yz} = F_{zy}$. Prove that $d(d\omega) = 0$.

3. Let $\omega = F(x,y,z)\,dx + G(x,y,z)\,dy + H(x,y,z)\,dz$ and assume that each of F, G, H satisfy the hypothesis of the preceding question. Prove that $d(d\omega) = 0$.

4. Use the preceding problem to prove that the differential 1-form

$$\omega = (3x^2 - 6yz)\,dx + (2y + 3xz)\,dy + (1 - 4xyz^2)\,dz$$

does not arise as the derivative $\omega = d\nu$ of any 0-form ν on $S = \mathbb{R}^3$.

5. For two differential 0-forms ν, ω prove that

$$d(\nu\omega) = (d\nu)\omega + \nu(d\omega).$$

6. For two differential 1-forms $\nu = A\,dx + B\,dy$, $\omega = C\,dx + D\,dy$ prove that

$$d(\nu \wedge \omega) = (d\nu) \wedge \omega - \nu \wedge (d\omega).$$

14 Stokes' Formula

1. Verify Stokes' Formula $\int_{\partial S} \omega = \int_S d\omega$ for $\omega = (2xy - x^2)\,dx + (x + y^2)\,dy$ and S the region in the xy-plane bounded by $y = x^2$ and $x^2 = y$. [See 10.6]

2. Verify Stokes' Formula $\int_{\partial S} \omega = \int_S d\omega$ for $\omega = (2x - z)\,dy \wedge dz + x^2 y\,dz \wedge dx - xz^2\,dx \wedge dy$ and S the region in $\mathbb{R}^3$ bounded by $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, $z = 1$. (This is a very long and tedious questions to answer!) [See 10.23]

3. Let S denote the region bounded by some ellipse (or other some other simple closed curve) in the xy-plane. Use Stokes' Formula to show that the area of S is given by $\int_{\partial S} x\,dy - y\,dx$. [See 10.8]

4. Calculate the area bounded by the ellipse $x = a\cos\theta$, $y = b\sin\theta$.

5. By considering an oriented 2-dimensional rectangle S in the xy-plane, explain how Stokes' formula $\int_{\partial S} \omega = \int_S d\omega$ leads to the definition of the derivative $d\omega$ of a differential 1-form $\omega = A\,dx + B\,dy$.

15 div, grad, curl

1. Consider the 0-form $\omega = (x^2 + y^2)/2$. Calculate the "gradient" 1-form $d\omega$ and sketch the corresponding vector field on $\mathbb{R}^2$.

2. Find a unit normal to the surface $S \subset \mathbb{R}^3$ defined by the equation

$$2x^2 + 4yz - 5z^2 = -10$$

at the pont $(3, -1, 2) \in S$. [See 7.37]

3. Consider the 1-form $\omega = -y\,dx + x\,dy$. Sketch the corresponding vector field. Then compute the "curl" 2-form $d\omega$. What feature of your sketch is captured by $d\omega$?

4. Consider the vector field $F = xz\mathbf{i} - y^2\mathbf{j} + 2x^2y\mathbf{k}$. Define curl$(F)$ in terms of the derivative of a 1-form and then calculate curl(F).

5. Consider the 0-form $\omega = x^2yz^3$ and the vector field $F = xz\mathbf{i} - y^2\mathbf{j} + 2x^2y\mathbf{k}$. Determine grad$(\omega)$, div$(F)$, curl$(F)$, div$(\phi F)$, div$(\omega F)$, curl$(\omega F)$. [See 7.34]