Consider \( f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases} \)

Show that \( f_x(0,0) \) exists, \( f_y(0,0) \) exist but that \( f(x,y) \) is not continuous at \( (0,0) \).

\[
\lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{0}{h^2} = 0
\]

\[
\lim_{h \to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h \to 0} \frac{0}{h^2} = 0
\]

Hence \( f_x(0,0) = 0 \), \( f_y(0,0) = 0 \) exist.

In last lecture we showed
\[
\lim_{x \to 0} f(x,y) = \frac{m}{1+n^2}
\]
Since this limit would depend on \( m \), it does not exist. Hence \( f(x, y) \) is not continuous at \((0,0)\).

**Fundamental Theorem of Calculus**

Let \( \omega \) be a 1-form on \( n \)-dimensional space.

Let \( S \) be a curve in \( \mathbb{R}^n \) from \( P \) to \( Q \).

**Theorem**

\[
\int_S d\omega = \int_S \omega
\]
Example Evaluate

$$I = \int_S (y^3 + 2x) \, dx + 3xy^2 \, dy$$

where $S$ is the straight line from $P=(0,0)$ to $Q=(1,2)$.

Solution (using above theorem)

Consider

$$\omega = x y^3 + x^2$$

Then

$$d\omega = (y^3 + 2x) \, dx + 3xy^2 \, dy$$

so

$$I = \int_S \omega = \omega \bigg|_Q - \omega \bigg|_P$$

$$= 9 - 0 = 9$$
The points \((x = t, y = 2t)\) trace out the line segment \(S\) as \(t\) goes from \(t = 0\) to \(t = 1\).

\[
\begin{align*}
x &= t \\
y &= 2t \\
dx &= dt \\
dy &= 2\, dt
\end{align*}
\]

\[
S = \int_0^1 (2t^3 + 2t)\, dt + 3t(2t)^2\, 2\, dt
\]

\[
= \int_0^1 32t^3 + 2t\, dt
\]

\[
= \left[ \frac{32}{4} t^4 + t^2 \right]_0^1 = 9.
\]
Problem: Evaluate
\[ I = \int_S (6xy^2 - y^3) \, dx + (6x^2y - 3xy^2) \, dy \]
where \( S \) is some curve from \( P = (1,2) \) to \( Q = (3,4) \).

Solution: Try to find \( \omega = F(x,y) \) such that
\[ d\omega = F_x \, dx + F_y \, dy \]
\[ = (6xy^2 - y^3) \, dx + (6x^2y - 3xy^2) \, dy \]
well
\[ F(x,y) = 3x^2y^2 - xy^3 + g(y) \]
\[ F(x,y) = 3x^2y^2 - xy^3 + h(x) \]
we conclude that \( g(y) \) and \( h(x) \) are equal to some constant \( c \),
\[ I = \int 3x^2y^2 - x y^3 + c \, ds \]

\[ = 3x^2y^2 - x y^3 + c \left( \frac{3}{4}, \frac{4}{3} \right) \]

\[ = \ldots \]

\[ = 236 \]