$$\boxed{\int_{\partial S} \omega = \int_S d\omega} \qquad (*)$$

next goal:

- define $d\omega$ for ω a 0-form
- give terminology for stating a hypothesis under which (*) holds

Differentiation of 0-forms

Given a 0-form

$$\omega = f(x, y, z)$$

we define the 1-form

$$\boxed{d\omega = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz}$$

We call $d\omega$ the exterior derivative of ω. We sometimes say that $d\omega$ is the total derivative of ω, or the

derivative of ω.

Example Find the exterior derivative of the ω-form

$$\omega = \sqrt{1-(x^2+y^2+z^2)}$$

on $S = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \leq 1\}$.

Solⁿ

$$d\omega = \frac{x}{\sqrt{1-(x^2+y^2+z^2)}}\,dx + \frac{y}{\sqrt{1-(x^2+y^2+z^2)}}\,dy$$

$$+ \frac{z}{\sqrt{1-(x^2+y^2+z^2)}}\,dz$$

Continuity, differentiability & partial derivatives

A function $f(x,y)$ is continuous if a small change in input only ever produces a small change in output.

More formally, $f(x,y)$ is continuous at a point (x_0,y_0) if for any $\varepsilon>0$ we can find $\delta>0$ such that $f(x,y)$ is defined and

$$|f(x,y)-f(x_0,y_0)|<\varepsilon$$

whenever $|x-x_0|<\delta$ and $|y-y_0|<\delta$.

Example Consider

$$f(x,y) = \begin{cases} 3xy & (x,y) \neq (1,2) \\ 0 & (x,y) = (1,2) \end{cases}$$

At the point $(1,2)$

$$\left.\begin{aligned} \lim_{(x,y)\to(1,2)} f(x,y) &= 6 \\ f(1,2) &= 0 \end{aligned}\right\}$$ means $f(x,y)$ is **not** continuous at $(1,2)$.

Example Consider

$$f(x,y) = \begin{cases} \dfrac{x^2-y^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Choose some constant m.

Suppose $x \to 0$. Then $y = mx \to 0$.

$$\lim_{(x,y)\to(0,0)} f(x,y) = ?$$

$$\lim_{\substack{x\to 0 \\ y=mx\to 0}} f(x,y) =$$

$$= \lim_{x\to 0} \frac{x^2 - (mx)^2}{x^2 + (mx)^2}$$

$$= \frac{1-m^2}{1+m^2}.$$

This answer depends on m.

Thus

$$\lim_{\substack{x\to 0 \\ y\to 0}} f(x,y) \text{ does } \underline{\text{not}} \text{ exist.}$$

It follows that $f(x,y)$ is not continuous at $(0,0)$.

<u>Definition</u> If a function $f(x, y)$ has continuous partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ in a region S, then f is said to be <u>continuously differentiable</u> in the region.

<u>Proposition</u> If f is continuously differentiable in a region then f is continuous in the region, and f is differentiable in the region.