Example: Evaluate

\[ L' = \int (x^2 - y) \, dx + (y^2 - x) \, dy \]

where \( S' \) is the line from \((0,1)\) to \((1,1)\) followed by the line from \((1,1)\) to \((1,2)\).

**Solution**

\[ L' = \int_{S_1} (x^2 - y) \, dx + (y^2 - x) \, dy + \int_{S_2} (x^2 - y) \, dx + (y^2 - x) \, dy \]

The line \( y = 1 \) contains \( S_1 \).

\[ \lim_{n \to \infty} \sum_{i=1}^{n} \left[ (x_i^2 - 1)(x_i - x_{i-1}) + (1^2 - x_i) \right] = 0 \]
\[ + \lim_{\| \mathbf{p} \| \to 0} \sum_{i=1}^{n} \left( (x_i^2 - 1) \cdot 0 + (y_i^2 - 1) \cdot (y_i - y_{i-1}) \right) \]

\[ = \int_{0}^{1} (x^2 - 1) \, dx + \int_{1}^{2} (y^2 - 1) \, dy \]

\[ \Rightarrow \frac{8}{3} \]

**Example:** Work is represented by the 1-form

\[ \omega = \left( 3x - 4y + 2z \right) \, dx \]

\[ + \left( 4x + 2y - 3z^2 \right) \, dy \]

\[ + \left( 2x^2 - 4y^2 + z^3 \right) \, dz \]

Find the work done in moving a particle once around the following ellipse in the xy-plane, in the anti-clockwise direction.

**Answer:**
\[
\text{work} = \int_S (3x - 4y) \, dx + (4x + 2y) \, dy
\]

\begin{align*}
x &= 4 \cos t \\
y &= 3 \sin t \\
dx &= -4 \sin t \, dt \\
dy &= 3 \cos t \, dt
\end{align*}

\[
\text{work} = \int_{2\pi}^{0} \left(3(4 \cos t) - 4(3 \sin t)\right)(-4 \sin t) \, dt + \\
+ \left(4(4 \cos t) + 2(3 \sin t)\right)3 \cos t \, dt
\]

\[
= \int_{0}^{2\pi} 48 - 30(\sin t)(\cos t) \, dt
\]
\[ = 48t - 15 \sin^2 t \int_0^{2\pi} \]

\[ = 90\pi \]
Partial Derivatives

Given a 0-form

\[ w = f(x, y, z) \]

we denote by

\[ \frac{df}{dx} \]

the 0-form obtained by regarding \( y \) and \( z \) as constants and differentiating \( w \) with respect to \( x \).

We call \( \frac{df}{dx} \) the partial derivative of \( f \) with respect to \( x \).
Example: Consider the function
\[ f(x, y, z) = \sqrt{1 - (x^2 + y^2 + z^2)} \]
defined on
\[ S = \{ (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1 \} \]
Calculate \( \frac{df}{dx} \).

Solution:
\[ f(x, y, z) = \left(1 - (x^2 + y^2 + z^2)\right)^{\frac{1}{2}} \]
\[ \frac{df}{dx} = \frac{1}{2} \left(1 - (x^2 + y^2 + z^2)\right)^{-\frac{1}{2}} (-2x) \]
\[ = \frac{-x}{\sqrt{1 - (x^2 + y^2 + z^2)}} \]
Similarly:

\[
\frac{df}{dy} = \frac{y}{\sqrt{1 - (x^2 + y^2 + z^2)}}
\]

\[
\frac{df}{dz} = \frac{z}{\sqrt{1 - (x^2 + y^2 + z^2)}}
\]

**Notation**
we often write

\[ f_x \]

in place of

\[ \frac{df}{dx} \]