Differential 1-forms in one variable

A <u>differential</u> <u>1-form</u> (henceforth a <u>1-form</u>) is a function of the type

$$\omega = f(x)\, h$$

which inputs numbers $x, h \in \mathbb{R}$ and returns the number $f(x)\, h$ (where $f(x)$ is some function).

<u>Example</u> Evaluate the 1-form

$$\omega = (x^2+6)\, h$$

at $x = 2$, $h = 0.5$.

<u>Soln</u> $(2^2+6)\, 0.5 = 5$.

<u>Notation</u>: we usually denote the 1-form

$$\omega = f(x)\, h$$

by

$$\omega = f(x)\, dx$$

Example Evaluate the 1-form

$$\omega = \sin(x)\, dx$$

at $x = \frac{\pi}{2}$, $dx = 0.25$

Sol^n $\sin\left(\frac{\pi}{2}\right) \times 0.25 = 0.25$

Defn Given a 1-form

$$\omega = f(x)\, dx$$

and an oriented interval $S = [a, b]$ we define the **integral**

$$\int_S \omega = \underbrace{\int_a^b f(x)\, dx}_{\text{Explained in first year.}}$$

Recall from 1st year

Informally $\int_a^b f(x)\,dx$ is the area between the curve $y = f(x)$ and the x-axis from a to b where, if $b > a$, areas above the x-axis are regarded as positive and areas below x-axis are negative.

Formally

$$\int_a^b f(x)\,dx = \lim_{\|P\| \to 0} \sum f(\xi_i)(x_i - x_{i-1})$$

where

- $\xi_i \in [x_{i-1}, x_i]$
- $P = \{a = x_0, x_1, x_2, \dots, x_n = b\}$ is a sequence of points in $[a, b]$.

$$- \|P\| = \max_{1 \leq i \leq n} |x_i - x_{i-1}|$$

Fundamental Theorem of Calculus

For a 0-form $\omega = F(x)$ defined on $S = [a, b]$ with $\frac{d}{dx}F(x) = f(x)$

we have

$$\boxed{\int_{\partial[a,b]} F(x) = \int_{[a,b]} f(x)\, dx}$$

or

$$\boxed{\int_{\partial S} \omega = \int_S d\omega}$$

Example Integrate the 1-form

$$\omega = (3x^2 + 2x)\, dx$$

on the oriented interval

$S = [3, 0]$.

Sol^n Consider the 0-form

$$F(x) = x^3 + x^2 .$$

Then $F'(x) = \underbrace{3x^2 + 2x}_{f(x)}$

So

$$\int_{\partial[3,0]} F(x) = \int_{[3,0]} f(x)\, dx = \int_{[3,0]} (3x^2+2x)\, dx$$

$$\|$$

$$F(0) - F(3) = (0^3 + 0^2) - (3^3 + 3^2)$$

$$= -36 .$$