Last lecture

Given
\[ F = F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k} \]

We can associate the work 1-form
\[ w = F_1 \, dx + F_2 \, dy + F_3 \, dz \]

and calculate
\[ dw = A \, dx \, dy + B \, dy \, dz + C \, dz \, dx \]

We define
\[ \text{Curl}(F) = B \hat{i} + C \hat{j} + A \hat{k} \]

Interpretation of curl

Imagine a vector field \( F \) on \( \mathbb{R}^3 \) describes the direction and speeds of particles in a fluid.

Place a rough small spherical ball in the fluid, with centre fixed at point \((x, y, z)\).
The fluid will cause the ball (with fixed centre) to rotate. The axis of the rotation is in the direction of the vector $\text{Curl}(\mathbf{F})$. The angular speed of rotation is given by the size of the vector $\text{Curl}(\mathbf{F})$.

**Example**

Consider

$$\mathbf{F}(x, y, z) = y \mathbf{i} - x \mathbf{j}.$$  

In the xy-plane we have ($\mathbf{z}=0$)
\[ \omega = y \, dx - x \, dy \]

\[ d\omega = dy \, dx - dx \, dy \]

\[ = -dx \, dy - dx \, dy \]

\[ = -2 \, dx \, dy \]

\[ \text{Curl}(\vec{F}) = -2 \frac{\hat{k}}{2} \]

so a ball rotate at speed 2, clockwise, about an axis parallel to the z-axis, no matter where we place the ball.

Divergence
Divergence

Given a vector field
\[ \mathbf{F} = F_1 \mathbf{i} + F_2 \mathbf{j} + F_3 \mathbf{k} \]
on \( \mathbb{R}^3 \) we define the associated flux 2-form

\[ \omega = F_3 \, dx \wedge dy + F_1 \, dy \wedge dz + F_2 \, dz \wedge dx \]

The exterior derivative of \( \omega \) is a 3-form

\[ d\omega = A \, dx \wedge dy \wedge dz \]

Definition: we define the divergence of \( \mathbf{F} \) to be the function

\[ \text{div} (\mathbf{F}) = A \]
Consider

\[ F = x^2 \mathbf{i} - y^2 \mathbf{j} + 2x^2 y \mathbf{k} \]

Let's find \( \text{div}(F) \).

\[ \omega = 2x^2 y \, dx \wedge dy + x^2 \, dy \wedge dz + \frac{y^2}{z} \, dz \wedge dx \]

\[ \text{div} \, \omega = \frac{\partial}{\partial z} x^2 - \frac{\partial}{\partial x} \left( \frac{y^2}{z} \right) \]

\[ = \frac{\partial}{\partial z} x^2 - \frac{-y^2}{z^2} \]

\[ = 2x^2 - \frac{y^2}{z^2} \]

So

\[ \text{div}(F) = 2x^2 - \frac{y^2}{z^2} \]
Let $P$ represent the flow of a fluid in $\mathbb{R}^3$. Place a small ball with centre fixed at $(x, y, z)$.

Fluid flows into the ball and out of the ball. The difference in measured by the number

\[ \text{div}(P)(x, y, z) \]