Notation for a function $F(x,y,z)$

we write

$$F_x = \frac{\partial}{\partial x} F, \quad F_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right)$$

Proposition Let $\omega = F(x,y,z)$ be a 0-form. suppose

$$F_{xy} = F_{yx}, \quad F_{xz} = F_{zx}, \quad F_{yz} = F_{zy}$$

Then

$$d(d\omega) = 0.$$

Proof

$$d(d\omega) = d(F_x dx + F_y dy + F_z dz)$$

$$= (F_{xx} dx + F_{xy} dy + F_{xz} dz) \wedge dx$$

$$+ (F_{yx} dx + F_{yy} dy + F_{yz} dz) \wedge dy$$

$$+ (F_{zz} dz + F_{yz} dy + F_{zx} dx) \wedge dz$$

$$= F_{xx} \underbrace{\cancel{dx \wedge dx}}_{=0} + F_{xy} \cancel{dy} \wedge dx + F_{xz} dz \wedge dx$$

$$+ F_{yx} \cancel{dx} \wedge dy + F_{yy} \underbrace{\cancel{dy \wedge dy}}_{=0} + F_{yz} dz \wedge dy$$

$$+ \cdots$$

$$= 0.$$

Example Prove that

$$\omega = (3x^2 - 6yz)\,dx + (2y + 3xz)\,dy + (1 - 4xyz^2)\,dz$$

does **not** arise as the exterior derivative of any 0-form v. That is, $\omega \neq dv$.

Sol^n It suffices to show that $d\omega \neq 0$.

$$d\omega = (6x\,dx - 6z\,dy - 6y\,dz) \wedge dx + (3z\,dx + 2\,dy + 3x\,dz) \wedge dy + (-4yz^2\,dx - 4xz^2\,dy - 8xyz\,dz) \wedge dz$$

$$= 6x\,dx \wedge dx - 6z\,dy \wedge dx - 6y\,dz \wedge dx + \dots$$

$$= \dots$$

$$\neq 0.$$

Differentiation of k-forms

A 2-form is an expression such as

$$\omega = A\,dx \wedge dy + B\,dy \wedge dz + C\,dz \wedge dx$$

where A, B, C are function in

$x, y, z, \ldots$

A 3-form is an expression such as

$$\omega = A\,dx \wedge dy \wedge dz + B\,dx \wedge dy \wedge dt + \ldots$$

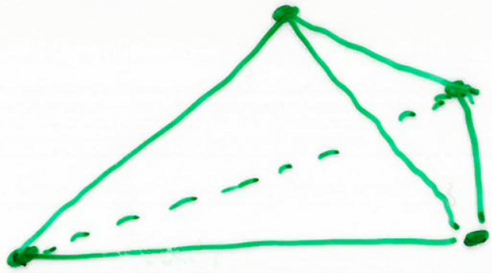
where $A, B, \ldots$ are functions of

$x, y, z, t, \ldots$

To understand integrals of 2-forms we needed to understand how to integrate constant 2-forms over oriented planar triangles.

(area)

For integrals of 3-forms we just need to understand how to integrate constant 3-forms over solid tetrahedra



An <u>orientation</u> of such a tetrahedron can be specified by an arrow on its surface pointing either inwards or outwards:

Given a 2-form ω can can define a 3-form

$d\omega$ such that

$$\int_{\partial S} \omega = \int_S d\omega$$

where S is an oriented 3-dimensional region.

The derivative $d\omega$ satisfies rules 1-6 from last lecture, and also:

7. $(dx \wedge dy) \wedge dz = dx \wedge (dy \wedge dz)$

We usually write

$$dx \wedge dy \wedge dz$$

<u>Exercise:</u> Calculate $d\omega$ for

$$\omega = x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy$$