Differentiation of 1-forms

For 1-forms ω and ω' and for 0-forms $A, B, C, \ldots$ in variables $x, y, z, \ldots$

1. $d(\omega + \omega') = d\omega + d\omega'$
2. $dA = \frac{\partial A}{\partial x} dx + \frac{\partial A}{\partial y} dy + \frac{\partial A}{\partial z} dz + \cdots$
3. $d(A\,dx + B\,dy + \cdots) =$
   $(dA) \wedge dx + (dB) \wedge dy + \cdots$
4. $dx \wedge dx = 0, \; dy \wedge dy = 0, \; \ldots$
5. $dx \wedge dy = -dy \wedge dx, \; \ldots$
6. $(\omega + \omega') \wedge dx = \omega \wedge dx + \omega' \wedge dy, \; \ldots$

<u>Example</u> calculate $d\omega$ for

$$\omega = xy\,dz + yz\,dx + zx\,dy.$$

Sol^n

$$d\omega = d(xy\,dz) + d(yz\,dx) + d(zx\,dy)$$

$$= d(xy)\wedge dz + d(yz)\wedge dx + d(zx)\wedge dy$$

$$= (y\,dx + x\,dy)\wedge dz + (z\,dy + y\,dz)\wedge dx + (x\,dz + z\,dx)\wedge dy$$

$$= y\,dx\wedge dz + x\,dy\wedge dz + z\,dy\wedge dx + y\,dz\wedge dx + x\,dz\wedge dy + z\,dx\wedge dy$$

$$= -y\,dz\wedge dx + x\,dy\wedge dz - z\,dx\wedge dy + y\,dz\wedge dx - x\,dy\wedge dz + z\,dx\wedge dy$$

$$= 0.$$

Last lecture we saw that rules 1-6 were enough to ensure:

for $\omega = A\,dx + B\,dy$

we

$$\boxed{d\omega = \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right) dx \wedge dy} \qquad (*)$$

To motivate rules 1-6 we'll explain why $(*)$ is precisely what is needed for Stokes' formula to hold.

So suppose

$$\omega = A\,dx + B\,dy$$

where A, B are functions of x, y.

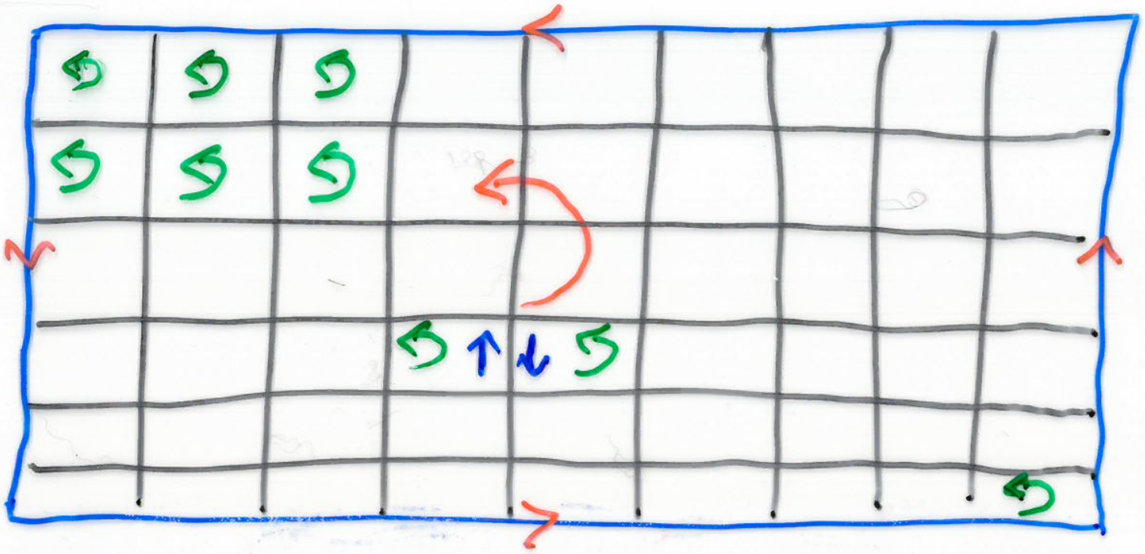
We want to <u>define</u>

$$d\omega = c\,dx \wedge dy$$

with c a function of x, y such that

$$\boxed{\int_{\partial S} A\,dx + B\,dy = \int_S c\,dx \wedge dy}$$

What does c have to be? For simplicity let's suppose that S is an oriented region in the xy-plane, with boundary ∂S oriented accordingly.



$$S = S_1 \cup S_2 \cup \dots \cup S_n$$

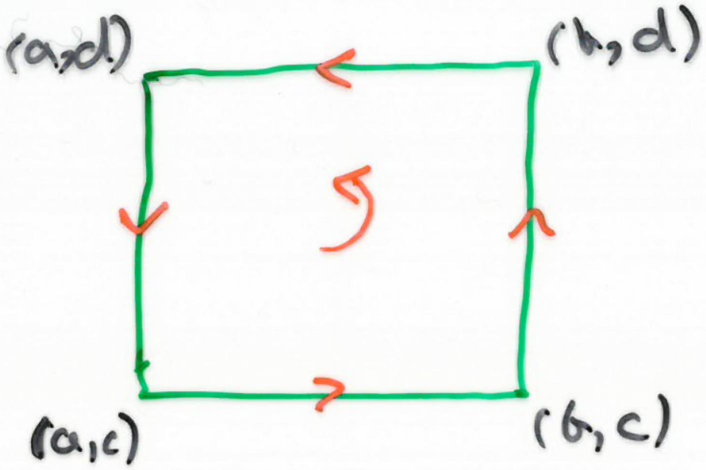
Note that

$$\int_{\partial S} A\,dx + B\,dy = \sum_{i=1}^{n} \int_{\partial S_i} A\,dx + B\,dy$$

so for each small S_i we just need

$$\boxed{\int_{\partial S_i} A\,dx + B\,dy = \int_{S_i} c\,dx \wedge dy}$$

Suppose S_i is the square



$$a \leq x \leq b$$

$$c \leq y \leq d$$

Assume a good $c\, dx \wedge dy$ does exist,

we have

$$\int_{\partial S_i} A\,dx + B\,dy$$

$$= \int_a^b A(x,c)\,dx + \int_c^d B(b,y)\,dy$$

$$+ \int_b^a A(x,d)\,dx + \int_d^c B(a,y)\,dy$$

$$= \int_c^d \left(B(b,y) - B(a,y) \right) dy$$

$$- \int_a^b \left(A(x,d) - A(x,c) \right) dx$$

$$\underset{FTC}{=} \int_c^d \left(\int_a^b \frac{\partial B}{\partial x}\, dx \right) dy$$

$$- \int_a^b \left(\int_c^d \frac{\partial A}{\partial y}\, dy \right) dx$$

$$= \int_{S_i} \frac{\partial B}{\partial x}\, dx \wedge dy - \int_{S_i} \frac{\partial A}{\partial y}\, dx \wedge dy$$

$$= \int_{S_i} \underbrace{\left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y} \right)}_{C} dx \wedge dy$$

Thus we need

$$d\omega = c\, dx \wedge dy = \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right) dx \wedge dy.$$