Differential of 1-forms

for 1-forms \( \omega \) and \( \omega' \) and for 0-forms \( A, B, C, \ldots \) in variables \( x, y, z, \ldots \):

1. \( d(\omega + \omega') = d\omega + d\omega' \)

2. \( dA = \frac{\partial A}{\partial x} dx + \frac{\partial A}{\partial y} dy + \frac{\partial A}{\partial z} dz + \cdots \)

3. \( d(A \, dx + B \, dy + \cdots) = \)

\[
(dA) \, dx + (dB) \, dy + \cdots
\]

4. \( dx \wedge dx = 0, \, dy \wedge dy = 0, \, \ldots \)

5. \( dx \wedge dy = -dy \wedge dx, \, \ldots \)

6. \( (\omega + \omega') \wedge dx = \omega \wedge dx + \omega' \wedge dy, \, \ldots \)

Example: Calculate \( d\omega \) for

\[ \omega = xy \, dz + yz \, dx + 2xz \, dy \]. \]
\[ dw = d(xy) + d(yz) + d(zx) \]
\[ = d(xy) + d(yz) + d(zx) \]
\[ = (ydx + xdy) + (zdy + ydz) + (xdz + zdx) \]
\[ = ydx + xdy + zdz + xdy + ydz + zdx \]
\[ = 0. \]
Last lecture we saw that rules 1-6 were enough to ensure:

\[ d\omega = (\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}) \, dx \wedge dy \]  

\[ \text{(\#)} \]

for \( \omega = A \, dx + B \, dy \)

We have

To motivate rules 1-6 we'll explain why (\#) is precisely what is needed for Stokes' formula to hold.
So suppose
\[ \omega = A \, dx + B \, dy \]
where \( A, B \) are functions of \( x, y \).

We want to define
\[ dw = C \, dx \wedge dy \]
with \( C \) a function of \( x, y \) such that
\[
\int_{\Gamma} A \, dx + B \, dy = \int_{\Sigma} C \, dx \wedge dy
\]

What does \( C \) have to be? For simplicity, let's suppose that \( S \) is an oriented region in the \( xy \)-plane, with boundary \( \partial S \) oriented accordingly.
\[ S = S_1 \cup S_2 \cup \ldots \cup S_n \]

Note that

\[ \int_A dx + Bdy = \sum_{i=1}^{n} \int_{\partial S_i} A dx + Bdy \]

So for each small \( S_i \) we just need

\[ \int_{\partial S_i} A dx + Bdy = \int_{S_i} C dx \land dy \]
Suppose $S_i$ is the square

$$(a, c) \quad (b, d)$$

$a \leq x \leq b$

$c \leq y \leq d$

Assume a good $c$ does exist.

We have

$$\int_A \, dx + B \, dy$$

$$ds_i = \int_a^b A(x, c) \, dx + \int_c^d B(b, y) \, dy$$

$$+ \int_a^b A(x, d) \, dx + \int_c^d B(a, y) \, dy$$
\[
\begin{align*}
&= \int_a^b (B(b,y) - B(a,y)) \, dy \\
&\quad - \int_a^b (\Phi(x, a) - \Phi(x, c)) \, dx \\
&= \int_c^d \left( \int_a^b \frac{\delta B}{\delta x} \, dx \right) \, dy \\
&\quad - \int_c^d \left( \int_a^b \frac{\delta A}{\delta y} \, dy \right) \, dx \\
&= \int_{S_2} \frac{\partial B}{\partial x} \, dx \, dy - \int_{S_2} \frac{\partial A}{\partial y} \, dx \, dy \\
&= \int_{S_2} \left( \frac{\partial B}{\partial x} - \frac{\partial A}{\partial y} \right) \, dx \, dy
\end{align*}
\]
Thus we need
\[ d\omega = c \, dx \, dy - (\frac{\partial F}{\partial x} - \frac{\partial G}{\partial y}) \, dx \, dy. \]