Partial derivatives of composite functions

If

$$u = F(x_1, x_2, \dots, x_n)$$

where

$$x_1 = g_1(r_1, r_2, \dots, r_p)$$

$$x_2 = g_2(r_1, r_2, \dots, r_p)$$

$$\vdots$$

$$x_n = g_n(r_1, r_2, \dots, r_p)$$

then

$$\frac{\partial u}{\partial r_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial r_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial r_i} + \dots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial r_i}$$

Proof is fairly straightforward.

Example Let

$$u = x^2 e^{yx}$$

$$x = t\cos(t)$$

$$y = t\sin(t)$$

Find

$$\frac{du}{dt} \quad \text{at} \quad t = \frac{\pi}{2}.$$

Sol^n

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

$$= \left(x^2 y e^{yx} + 2e^{yx}x\right)\left(\cos(t) - t\sin(t)\right)$$

$$+$$

$$x^3 e^{yx}\left(\sin(t) + t\cos(t)\right)$$

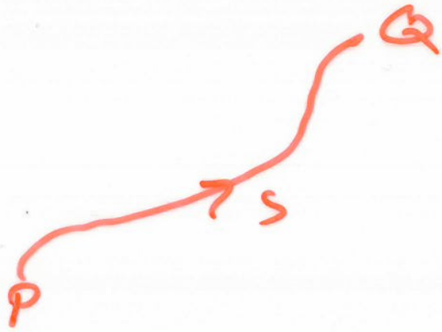
Evaluate at $t = \frac{\pi}{2}$

get $\frac{\partial u}{\partial t} = 0.$

Proof of fundamental theorem of calculus $(n=2)$

We want to prove

$$\int_S d\omega = \int_{\partial S} \omega$$



Proof

suppose $\omega = F(x,y)$

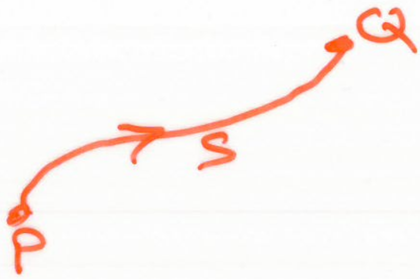
suppose $x = g(t)$, $y = h(t)$ is a parametrization of S as t varies from $t=t_0$ to $t=t_1$.

$$\int_S d\omega = \int_S F_x(x,y)\,dx + F_y(x,y)\,dy$$

$$= \int_{t_0}^{t_1} \frac{\partial F}{\partial x}(g(t), h(t))\, g'(t)\, dt + \frac{\partial F}{\partial y}(g(t), h(t))\, h'(t)\, dt$$

$$= \int_{t_0}^{t_1} \left(\frac{\partial F}{\partial x}(g(t), h(t))\, g'(t) + \frac{\partial F}{\partial y}(g(t), h(t))\, h'(t) \right) dt$$

$$= \int_{t_0}^{t_1} \left(\frac{\partial F}{\partial t} \right) dt$$



$$\underset{\text{FTC one variable}}{=} F(g(t_1), h(t_1)) - F(g(t_0), h(t_0))$$

$$= F(Q) - F(P).$$

QED

$$= \int_{\partial S} \omega .$$

Summary of 1-forms and an introduction to 2-forms

- A 1-form is an expression such as

$$\omega = A\,dx + B\,dy$$

that can be integrated over oriented curves

- A 2-form is an expression such as

$$\omega = A\,dx \wedge dy + B\,dy \wedge dz + C\,dz \wedge dx$$

that can be "integrated" over "2-dimensional oriented regions"

- Integrals of 1-forms are just limits of sums of integrals of constant 1-forms over oriented straight-line segments.

- Integrals of 2-forms are just limits of sums of "integrals" of "constant 2-forms" over "oriented planar triangles."