Example Consider

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Show that $f_x(0,0)$ exists, $f_y(0,0)$ exists but that $f(x,y)$ is **not** continuous at $(0,0)$.

Sol^n

$$\lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{0}{h} = 0$$

$$\lim_{h \to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h \to 0} \frac{0}{h} = 0.$$

Hence $f_x(0,0) = 0$, $f_y(0,0) = 0$ exist.

Last lecture we saw

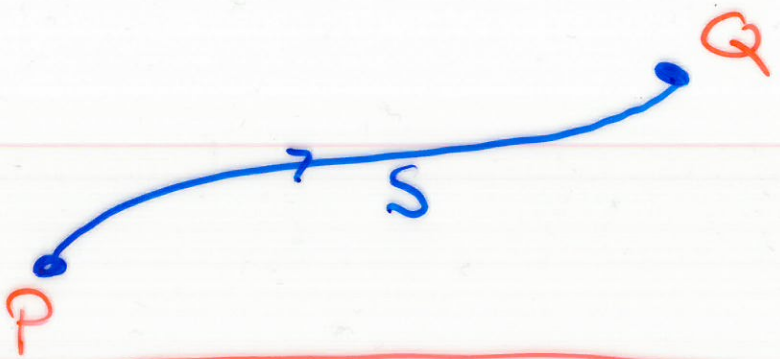
$$\lim_{\substack{x \to 0 \\ y = mx \to 0}} f(x,y) = \frac{m}{1+m^2}.$$

Since this limit depends on m, it doesn't exist. Hence $f(x,y)$ is not continuous at $(0,0)$.

Fundamental Theorem of Calculus again!

Let ω be a 0-form on n-dimensional space.

Let S be a curve in $\mathbb{R}^n$ from P to Q.



Theorem

$$\boxed{\int_S d\omega = \int_{\partial S} \omega}$$

Example Evaluate

$$I = \int_S (y^3 + 2x)\,dx + 3xy^2\,dy$$

where S is the straight line from $P=(0,0)$ to $Q=(1,2)$.

Sol^n (using above theorem)

Consider

$$\omega = xy^3 + x^2$$

Then

$$d\omega = (y^3+2x)\,dx + (3xy^2)\,dy$$

So

$$I = \int_{\partial S} \omega = \omega\big|_Q - \omega\big|_P$$

$$= 9 - 0 = 9.$$

$\underline{\text{Sol}^n}$ (Using the definition of an integral)

The points

$$(x = t,\ y = 2t)$$

trace out the line segment S as t goes from $t=0$, to $t=1$.

$$x = t \qquad\qquad y = 2t$$

$$dx = dt \qquad\qquad dy = 2\,dt$$

$$I = \int_{t=0}^{1} \left((2t)^3 + 2t\right) + 3t\,(2t)^2 . 2 \quad dt$$

$$= \int_0^1 32t^3 + 2t \ \ dt$$

$$= \frac{32t^4}{4} + t^2 \Big|_0^1 = 9.$$

Problem Evaluate

$$I = \int_S (6xy^2 - y^3)\,dx + (6x^2y - 3xy^2)\,dy$$

where S is some curve

from $P = (1,2)$ to $Q = (3,4)$.

Sol^n Try to find $\omega = F(x,y)$

such that

$$d\omega = F_x\,dx + F_y\,dy$$

$$= (6xy^2 - y^3)\,dx + (6x^2y - 3xy^2)\,dy.$$

Well

$$F(x,y) = 3x^2y^2 - xy^3 + g(y)$$

$$F(x,y) = 3x^2y^2 - xy^3 + h(x)$$

We conclude that $g(y)$ and

$h(x)$ are equal to some constant c.

So

$$I = \int_S d\left(3x^2y^2 - xy^3 + c\right)$$

$$\underset{FTC}{=} \int_{\partial S} 3x^2y^2 - xy^3 + c$$

$$= 3x^2y^2 - xy^3 + c \Big|_{(1,2)}^{(3,4)}$$

$$= \ldots$$

$$= 236 .$$