Differentiation of 0-forms

Given a 0-form

$$\omega = f(x, y, z)$$

we define the 1-form

$$\boxed{d\omega = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz}$$

we call $d\omega$ the exterior derivative of ω. Sometimes we say that $d\omega$ is the total derivative of ω, or the differential of ω.

Example Find the exterior derivative of the 0-form

$$\omega = \sqrt{1-(x^2+y^2+z^2)}$$

on $S = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \leq 1\}$.

Sol 4

$$dw = \frac{x}{\sqrt{x^2+y^2+z^2-1}}\,dx + \frac{y}{\sqrt{x^2+y^2+z^2-1}}\,dy$$

$$+ \frac{z}{\sqrt{x^2+y^2+z^2-1}}\,dx$$

Continuity, differentiability & partial derivatives

A function $f(x,y)$ is continuous if a small change in input only ever produces a small change in output.

More formally, $f(x,y)$ is continuous at a point (x_0, y_0) if for any $\varepsilon > 0$ we can find a $\delta > 0$ such that $f(x,y)$ is defined and

$$|f(x,y) - f(x_0, y_0)| < \varepsilon$$

whenever $|x - x_0| < \delta$ and $|y - y_0| < \delta$.

<u>Example</u> Consider

$$f(x,y) = \begin{cases} 3xy, & (x,y) \neq (1,2) \\ 0, & (x,y) = (1,2) \end{cases}$$

At the point $(1,2)$

$$\left.\begin{aligned} \lim_{(x,y)\to(1,2)} f(x,y) &= 6 \\ f(1,2) &= 0 \end{aligned}\right\}$$

Means $f(x,y)$ is <u>not</u> continuous at $(1,2)$.

Example Consider

$$f(x,y) = \begin{cases} \dfrac{x^2-y^2}{x^2+y^2} & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$$

Choose some constant m.

Suppose $x \to 0$. Then $y = mx \to 0$.

$$\lim_{(x,y)\to(0,0)} f(x,y) = ?$$

$$\lim_{\substack{x\to 0 \\ y=mx\to 0}} f(x,y) =$$

$$= \lim_{x\to 0} \frac{x^2-m^2x^2}{x^2+m^2x^2}$$

$$= \frac{1-m^2}{1+m^2}.$$

The answer depends on m.

Then

$$\lim_{\substack{x \to 0 \\ y \to 0}} f(x,y) \quad \text{does } \underline{\text{not}} \text{ exist.}$$

It follows that $f(x,y)$ is not continuous at $(0,0)$.

<u>Definition</u> If a function $f(x,y)$ has continuous partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ in a region S, then f is said to be <u>continuously differentiable</u> in the region.

Proposition If f is continuously differentiable in a region then f is continuous in the region, and f is differentiable in the region.