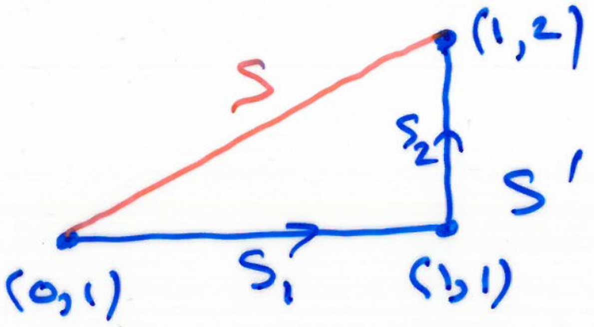
Example Evaluate

$$L' = \int_{S'} (x^2-y)\,dx + (y^2-x)\,dy$$

where S' is the line from $(0,1)$ to $(1,1)$ followed by the line from $(1,1)$ to $(1,2)$.

Sol^n



$$L' = \int_{S_1} (x^2-y)\,dx + (y^2-x)\,dy + \int_{S_2} (x^2-y)\,dx + (y^2-x)\,dy$$

The line $y=1$ contains S_1

The line $x=1$ contains S_2

$$= \lim_{\|P\|\to 0} \sum_{i=1}^{n} (x_i^2-1)(x_i - x_{i-1}) + \cancel{(1-x_i)}\,0$$

$$+ \lim_{\|P\| \to 0} \sum_{i=1}^{n} \cancel{(1^2 - y_i)} . 0 + (y_i^2 - 1) . (y_i - y_{i-1})$$

$$= \int_0^1 x^2 - 1 \, . \, dx + \int_1^2 y^2 - 1 \, . \, dy$$

$= \ldots$

$= \frac{8}{3}$

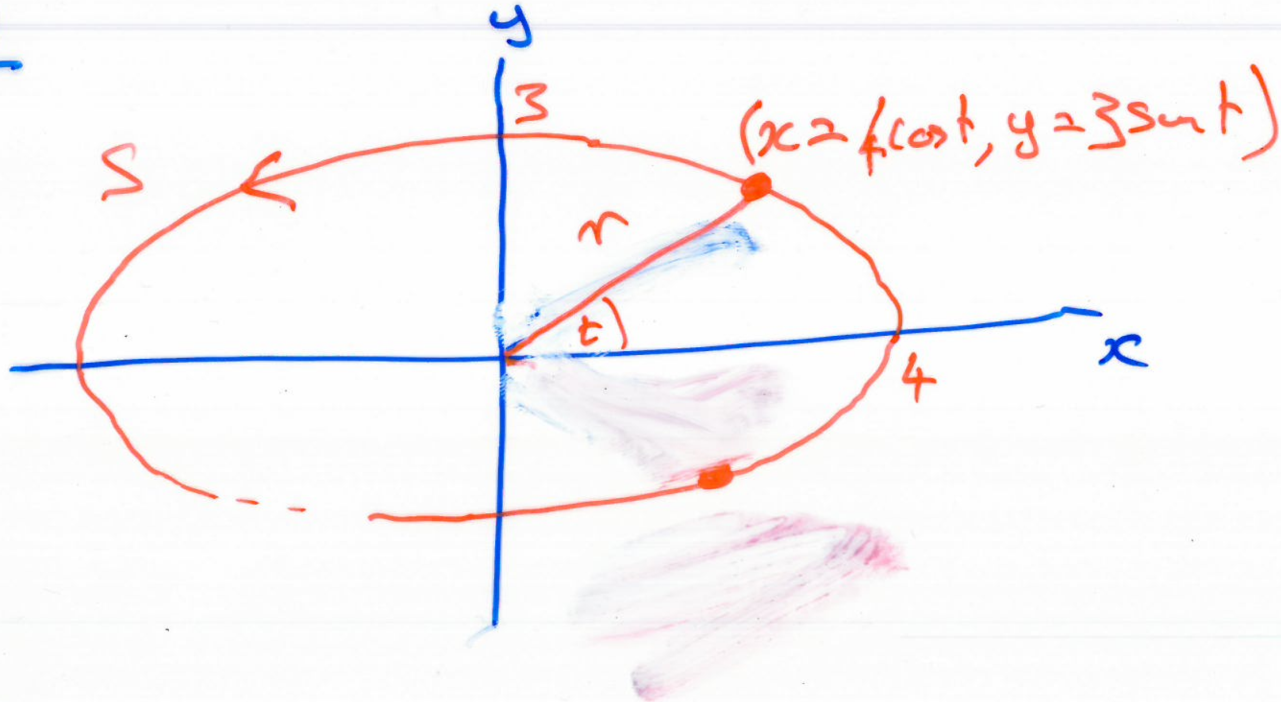
<u>Example</u> work is represented by the 1-form

$$\omega = (3x - 4y + 2z)\,dx + (4x + 2y - 3z^2)\,dy + (2xz - 4y^2 + z^3)\,dz$$

Find the work done in moving a particle once, in the anti-

clockwise direction, around an ellipse C in the xy-plane with centre at the origin and semi-major and semi-minor axes ~~...~~ $x=4$, $y=3$.

Soln



Work =

$$\int_S (3x-4y)\,dx + (4x+2y)\,dy$$

$x = 4\cos t$

$y = 3\sin t$

$dx = -4\sin t\,dt$

$dy = 3\cos t\,dt$

$$= \int_{t=0}^{2\pi} \Big(3(4\cos t) - 4(3\sin t)\Big)\Big(-4\sin t\, dt\Big)$$

$$+ \Big(4(4\cos t) + 2(3\sin t)\Big)\Big(3\cos t\, dt\Big)$$

$= \cdots$

$$= \int_0^{2\pi} (48 - 30\sin t \cdot \cos t)\, dt$$

$$= 48t - 15\sin^2 t \Big|_0^{2\pi} = 96\pi$$

Partial Derivatives

Given a 0-form

$$\omega = f(x, y, z)$$

we denote by

$$\frac{\partial f}{\partial x}$$

the 0-form got by regarding y and z as constants and differentiating with respect to x.

We call $\frac{\partial f}{\partial x}$ the partial derivative of f with respect to x.

Example Consider the function

$$f(x,y,z) = \sqrt{1-(x^2+y^2+z^2)}$$

defined on

$$S = \{(x,y,z) : x^2+y^2+z^2 = 1\}$$

Calculate $\frac{\partial f}{\partial x}$.

Sol^n

$$f(x,y,z) = (1-(x^2+y^2+z^2))^{\frac{1}{2}}$$

$$\frac{\partial f}{\partial x} = \frac{1}{2}\left(1-(x^2+y^2+z^2)\right)^{-\frac{1}{2}}(-2x)$$

$$= \frac{x}{\sqrt{x^2+y^2+z^2-1}}$$

Similarly:

$$\frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2+y^2+z^2-1}}$$

$$\frac{\partial f}{\partial z} = \frac{z}{\sqrt{x^2+y^2+z^2-1}}$$

NOTATION We often write

f_x in place of $\frac{\partial f}{\partial x}$.