<u>Example</u> Find the 1-form

$$\omega = A\,dx + B\,dy + C\,dz$$

describing the function "work" in the constant force field where displacement of a particle

from $(0,0,0)$ to $(4,0,0)$ needs 3 units of work
" $(1,-1,0)$ " $(1,1,0)$ " 2 " "
" $(0,0,0)$ " $(3,0,2)$ " 5 " "

<u>Sol^n</u>

$$\left.\begin{aligned} A.4 &= 3 \\ B.2 &= 2 \\ A.3 + C.2 &= 5 \end{aligned}\right\}$$

$A = \frac{3}{4}$
$B = 1$
$C = \frac{11}{8}$

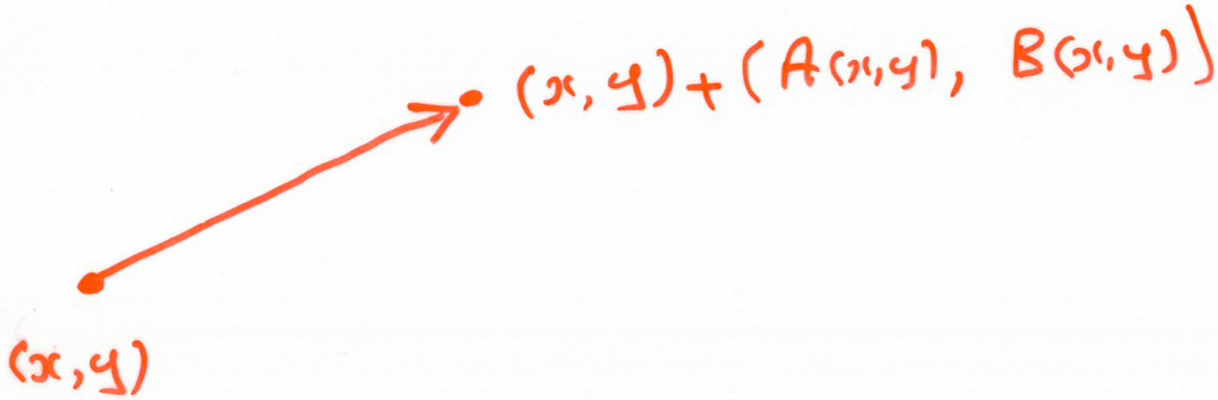
So

$$\omega = \frac{3}{4}dx + dy + \frac{11}{8}dz$$

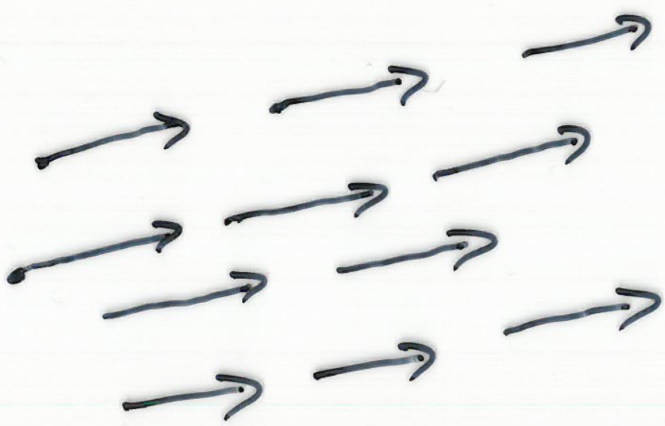
We can think of a 1-form

$$\omega = A(x,y)\,dx + B(x,y)\,dy$$

as a collection of arrows in the plane. For each point (x,y) in the plane we have the arrow



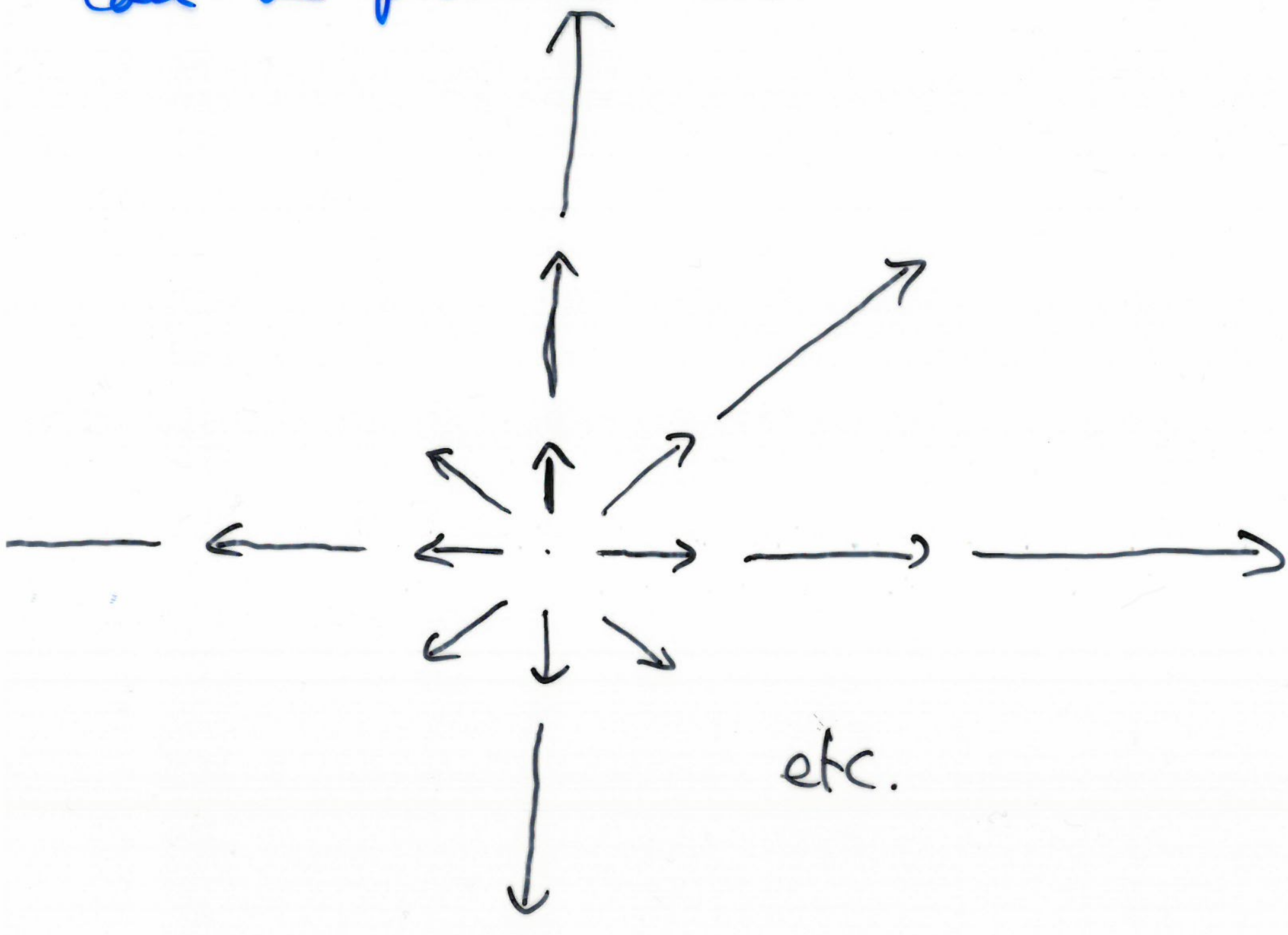
<u>Example</u> The 1-form $\omega = 2dx + dy$ can be pictured as



Example The 1-form

$$\omega = x\,dx + y\,dy$$

can be pictured as



Integration of 1-forms

Let $\omega = A(x,y)\,dx + B(x,y)\,dy$
be a 1-form.

Let $S \subseteq \mathbb{R}^2$ be a 1-dimensional oriented, connected subset.

Informally if we think of ω as "work done" then

$$\int_S A(x,y)\,dx + B(x,y)\,dy$$

is the total work done in moving

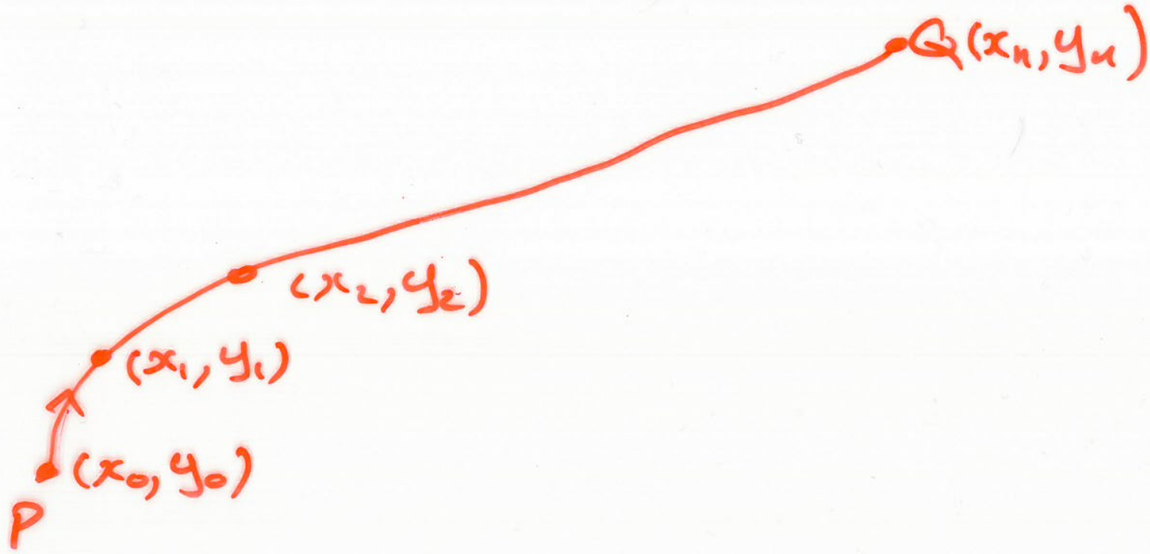
the particle from P to Q

along S.

Formally

$$\int_S A(x,y)\,dx + B(x,y)\,dy =$$

$$\lim_{\|P\| \to 0} \sum_{i=1}^{n} A(x_i;y_i).(x_i - x_{i-1}) + B(x_i;y_i).(y_i - y_{i-1})$$



where :

- $P = \{(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)\}$

is a sequence of points on S, with (x_0, y_0) the initial point of S, and (x_n, y_n) the final point of S,

- $$\|P\| = \max_{1 \le i \le n} \|(x_i, y_i) - (x_{i-1}, y_{i-1})\|$$

with $\|(x, y)\| = \sqrt{x^2 + y^2}$

Example Let S be the line segment from $(0,1)$ to $(1,2)$. Evaluate

$$L = \int_S (x^2-y)\,dx + (y^2+x)\,dy .$$

Sol^n The line $y = x+1$ passes through $(0,1)$ and $(1,2)$.

so $P = \{(x_0,y_0),(x_1,y_1),\dots,(x_n,y_n)\}$ has the form

$$P = \{(x_0,x_0+1),(x_1,x_1+1),(x_2,x_2+1),\dots,(x_n,x_n+1)\}$$

$$L = \lim_{\|P\| \to 0} \sum_{i=1}^{n} (x_i^2 - x_i - 1)(x_i - x_{i-1})$$

$$+ \left((x_i + 1)^2 + x_i\right)(x_i - x_{i-1})$$

$$= \int_0^1 \left(x^2 - x - 1 + x^2 + 2x + 1 + x\right) \, dx$$

$$= \int_0^1 2x^2 + 2x \quad dx$$

$$\vdots$$

$$= \frac{5}{3} \, .$$