Proof of the fundamental theorem of calculus

Suppose that the Galway to Dublin train has a functioning speedometer but a broken mileometer. To estimate the distance travelled between the time $t=a$ and $t=b$ the driver could calculate

$$\sum_{i=1}^{n} f(t_i)(t_i - t_{i-1})$$

where $f(t)$ = speed of the train at time t, and

$$a = t_0 < t_1 < t_2 < \cdots < t_n = a\,.$$

Let $F(t)$ = the total distance travelled by time t.

So

$$f(t) = F'(t)$$

and roughly

$$F(b) - F(a) \simeq \sum_{i=1}^{n} f(t_i)(t_i - t_{i-1})$$

Taking limits "as $n \to \infty$"

$$F(t) - F(a) = \lim_{n \to \infty} \sum_{i=1}^{n} f(t_i)(t_i - t_{i-1})$$

$$\boxed{F(b) - F(a) = \int_a^b f(t)\,dt}$$

Fundamental Theorem of Calculus.

Definition Given a 0-form $\omega = F(x)$ we define its **derivative** to be the 1-form

$$d\omega = F'(x)\,dx$$

Recall from 1st that substitutions can be useful for calculating integrals.

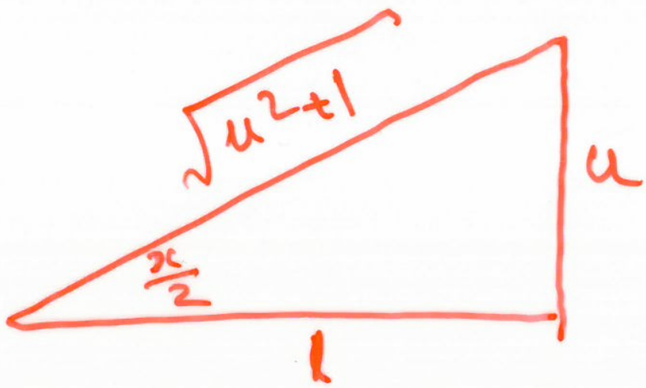
Example Find a differential 0-form ω whose derivative $d\omega$ is

$$d\omega = \frac{1}{5+3\cos(x)}\,dx$$

Sol^n Using the language of 1st year maths, we want to find

$$w = \int \frac{1}{5+3\cos(x)}\,dx$$ indefinite integral

Let $u = \tan\left(\frac{x}{2}\right)$



$$\sin\left(\frac{x}{2}\right) = \frac{u}{\sqrt{u^2+1}} \qquad \cos\left(\frac{x}{2}\right) = \frac{1}{\sqrt{u^2+1}}$$

$$\cos(x) = \cos\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) - \sin\left(\frac{x}{2}\right)\sin\left(\frac{x}{2}\right)$$

$$= \frac{1}{u^2+1} - \frac{u^2}{u^2+1}$$

$$= \frac{1-u^2}{1+u^2}$$

$$du = \frac{1}{2}\sec^2\left(\frac{x}{2}\right) dx$$

and

$$dx = 2\cos^2\left(\frac{x}{2}\right) du = \frac{2}{u^2+1} du$$

So

$$w = \int \frac{1}{5+3\cos(x)} dx$$

$$= \int \left(\frac{2}{u^2+1}\right)\left(\frac{1}{5+3\left(\frac{1-u^2}{1+u^2}\right)}\right) du$$

$\vdots$

$$= \int \frac{du}{4+u^2}$$

from log book:

$$w = \frac{1}{2}\tan^{-1}\frac{u}{2} + C$$

$$= \frac{1}{2}\tan^{-1}\left\{\frac{1}{2}\tan\left(\frac{x}{2}\right)\right\} + C.$$

Take $c = 0$ say. Then

$$w = \frac{1}{2} \tan^{-1}\left(\frac{1}{2} \tan\left(\frac{x}{2}\right)\right)$$

has derivative

$$dw = \frac{1}{5 + 3\cos(x)} \, dx .$$