Differential 1-forms in one variable

A differential 1-form (henceforth a 1-form) is a function of the type

$$\omega = f(x)\, h$$

which inputs two numbers $x, h \in \mathbb{R}$ and returns the number $f(x)\, h$.

Example Evaluate the 1-form

$$\omega = (x^2+6)\, h$$

at $x=2$, $h=0.5$.

Sol^n $(2^2+6)\, 0.5 = 5$

Notation We usually denote the 1-form

$$\omega = f(x)\, h$$

by

$$\omega = f(x)\, dx$$

Example Evaluate the 1-form

$$\omega = \sin(x)\,dx$$

at $x = \frac{\pi}{2}$, $h = 0.25$

Soln $\sin\left(\frac{\pi}{2}\right) 0.25 = 0.25$

Defn Given a 1-form

$$\omega = f(x)\,dx$$

and an oriented interval $S = [a, b]$ we define the integral

$$\int_S \omega = \underbrace{\int_a^b f(x)\,dx}_{\text{Explained in First Year.}}$$

Recall from 1st Year:

Informally $\int_a^b f(x)\,dx$ is the area between the curve $y=f(x)$ and the x-axis from a to b, where, if $b>a$, areas above the x-axis are regarded as positive and areas ~~th~~ below x-axis are negative.

Formally:

$$\int_a^b f(x)\,dx = \lim_{\|P\|\to 0} \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1})$$

where

- $\xi_i \in [x_{i-1}, x_i]$
- $P = \{a = x_0, x_1, x_2, \dots, x_n = b\}$ is a sequence of points in $[a, b]$. (partition!)

$$- \|P\| = \max_{1 \leq i \leq n} |x_i - x_{i-1}|$$

Fundamental Theorem of Calculus

For a 0-form $\omega = F(x)$ defined on $S = [a, b]$ with $\frac{d}{dx} F(x) = f(x)$

we have

$$\boxed{\int_{\partial[a,b]} F(x) = \int_{[a,b]} f(x)\, dx}$$

or

$$\boxed{\int_{\partial S} \omega = \int_{S} \partial\omega \quad \text{where } \partial\omega := f(x)\, dx}$$

Example Integrate the 1-form $\omega = (3x^2 + 2x)\,dx$ on the oriented interval $S = [3,0]$.

Sol^n Consider the 0-form

$$F(x) = x^3 + x^2.$$

Then $F'(x) = 3x^2 + 2x$

So

$$\int_{\partial[3,0]} F(x) = \int_{[3,0]} f(x)\,dx = \int_{[3,0]} (3x^2+2x)\,dx$$

$$\parallel$$

$$F(0) - F(3) = (0^3+0^2) - (3^3+3^2)$$

$$= -36.$$