Last lecture

Given

$$F = F_1 \underline{\underline{i}} + F_2 \underline{\underline{j}} + F_3 \underline{\underline{k}}$$

we consider the associated work 1-form

$$\omega = F_1\, dx + F_2\, dy + F_3\, dz$$

and calculate

$$d\omega = A\, dx \wedge dy + B\, dy \wedge dz + C\, dz \wedge dx.$$

we define

$$\mathrm{Curl}(F) = B \underline{\underline{i}} + C \underline{\underline{j}} + A \underline{\underline{k}}.$$

Interpretation of curl

Imagine a vector field F on $\mathbb{R}^3$ describes the directions and speeds of particles in a fluid.

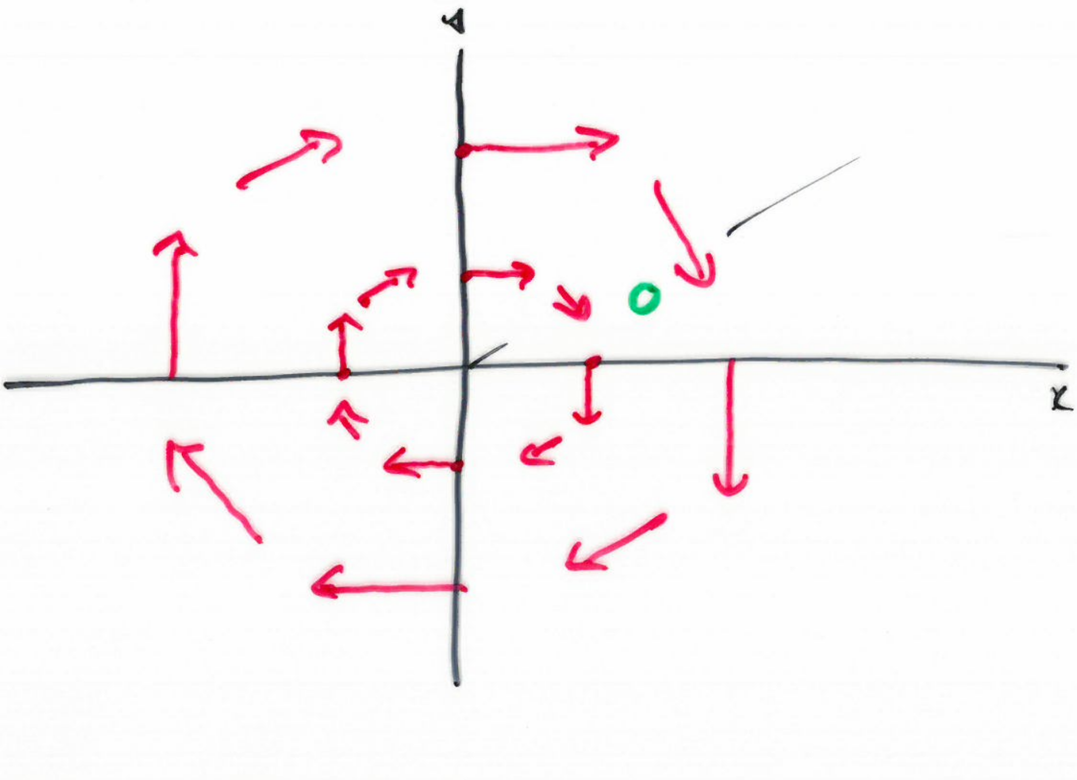
Place a small rough spherical ball in the fluid, with centre fixed at point (x, y, z).

The fluid will make the ball rotate. The axis of rotation is the direction of the vector curl(F). The angular speed of rotation is given by the size of the vector curl(F).

Example Consider

$$F(x,y,z) = y\,\underline{i} - x\,\underline{j}$$

in the xy-plane we have



$$\omega = y\,dx - x\,dy$$

$$d\omega = dy \wedge dx - dx \wedge dy$$

$$= -2\,dx \wedge dy$$

$$\text{curl}(F) = -2\,\underline{\underline{k}}$$

so a ball rotates at speed 2, clockwise, about an axis parallel to the z-axes, no matter where we place the ball.

<u>Divergence</u>

Given a vector field

$$F = F_1\,\underline{\underline{i}} + F_2\,\underline{\underline{j}} + F_3\,\underline{\underline{k}}$$

on $\mathbb{R}^3$ we define the associated

<u>flux 2-form</u>

$$\omega = F_3\,dx \wedge dy + F_1\,dy \wedge dz + F_2\,dz \wedge dx$$

The exterior derivative of ω is a 3-form

$$d\omega = A \, dx \wedge dy \wedge dz$$

for some function A.

<u>Definition</u> We define the <u>divergence</u> of F to be the function

$$\operatorname{div}(F) = A .$$

<u>Example</u> $F = xz \, \underline{\underline{i}} - y^2 \, \underline{\underline{j}} + 2x^2y \, \underline{\underline{k}}$

$$\omega = 2x^2y \, dx \wedge dy + xz \, dy \wedge dz - y^2 \, dz \wedge dx$$

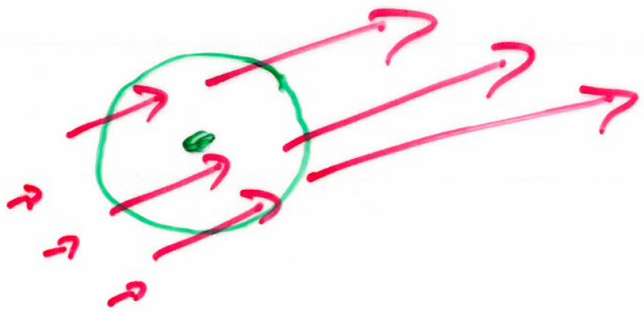
$$d\omega = z \, dx \wedge dy \wedge dz - 2y \, dy \wedge dz \wedge dx$$

$$= (z - 2y) \, dx \wedge dy \wedge dz$$

So

$$\operatorname{div}(F) = z - 2y .$$

<u>Interpretation</u> Let F represent the flow of a fluid in $\mathbb{R}^3$. Place a small ball at (x, y, z).



Fluid flows into the ball and out from the ball. The difference is measured by the number

$$\operatorname{div}(F)(x, y, z) .$$