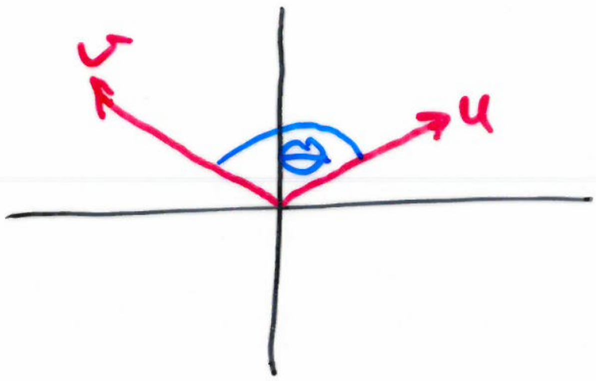
Dot products of vectors

Given two vectors

$$u = (u_1, u_2) \in \mathbb{R}^2$$
$$v = (v_1, v_2) \in \mathbb{R}^2$$

in the plane



we define the dot product to be the number

$$u \cdot v = u_1 v_1 + u_2 v_2$$

Example If $u = (2, 3)$, $v = (4, 5)$ then

$$u \cdot v = 2 \cdot 4 + 3 \cdot 5 = 23$$

We define the length of u to be

$$|u| = \sqrt{u_1^2 + u_2^2}$$

It is easy to prove:

Theorem $u.v = |u|\,|v| \cos(\theta)$

In particular, u and v are perpendicular to each other if and only if $u.v = 0$

For two vectors

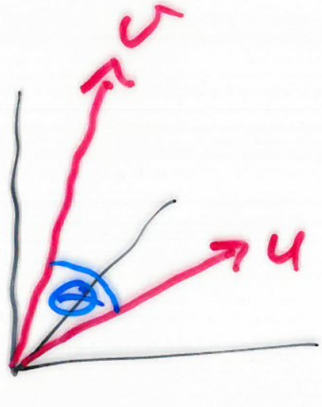
$$u = (u_1, u_2, u_3) \in \mathbb{R}^3$$

$$v = (v_1, v_2, v_3) \in \mathbb{R}^3$$

we define the dot product

$$u.v = u_1 v_1 + u_2 v_2 + u_3 v_3$$

Again: $|u| = \sqrt{u_1^2 + u_2^2 + u_3^2}$



Theorem

$u.v = |u|\,|v| \cos(\theta)$

In particular, u, v are right-angles if and only if

$u.v = 0$

Div, Grad, Curl and all that

Gradient

Let $\phi(x,y,z)$ be a real-valued differentiable function. The **gradient** of ϕ is defined as

$$\operatorname{grad}\phi = \nabla\phi = \frac{\partial\phi}{\partial x}\underline{\underline{i}} + \frac{\partial\phi}{\partial y}\underline{\underline{j}} + \frac{\partial\phi}{\partial z}\underline{\underline{k}}$$

where

$$\underline{\underline{i}} = (1,0,0)$$

$$\underline{\underline{j}} = (0,1,0)$$

$$\underline{\underline{k}} = (0,0,1)$$

Example $\phi(x,y,z) = x^2+y^2+z^2$

$$\operatorname{grad}\phi = 2x\,\underline{\underline{i}} + 2y\,\underline{\underline{j}} + 2z\,\underline{\underline{k}}$$

$$= (2x, 2y, 2z).$$

We pronounce $\nabla \phi$ as "del phi"

We could also think of the gradient ~~[illegible]~~ of a 0-form $\omega = \phi$ as the derivative

$$\nabla \phi = d\phi$$

<u>Interpretation of the gradient (or exterior derivative of 0-form)</u>

Consider a surface S defined by an equation

$$\phi(x, y, z) = k \quad (k \text{ a constant})$$

<u>Example</u> Let $\phi(x,y,z) = x^2+y^2+z^2$
Let $k = 9$

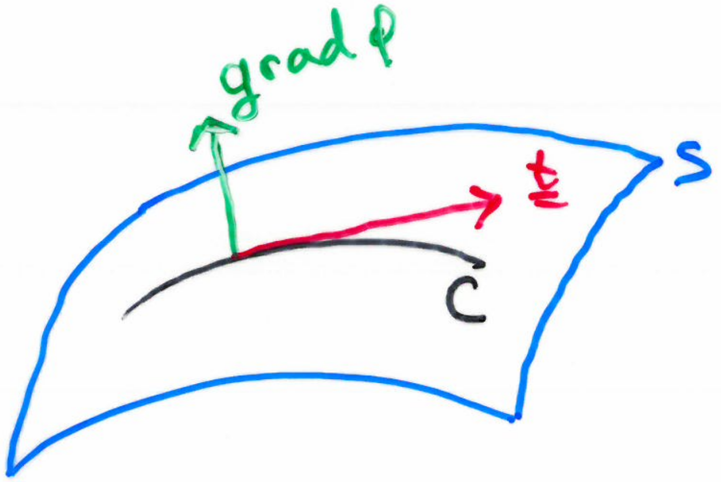
The equation

$$x^2+y^2+z^2 = 9$$

describes the sphere of radius 3, centred at the origin.

Let C be a curve on our surface S parametrized on

$$C : \mathbb{R} \longrightarrow S, \quad t \longmapsto (x(t), y(t), z(t)) .$$



Note that

$$\phi(x(t), y(t), z(t)) = k$$

k the constant, whatever curve C we choose.

The chain rule gives:

$$0 = \frac{d\phi}{dt} = \frac{\partial\phi}{\partial x}\frac{dx}{dt} + \frac{\partial\phi}{\partial y}\frac{dy}{dt} + \frac{\partial\phi}{\partial z}\frac{dz}{dt}$$

$$0 = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right) \cdot \left(\frac{\partial x}{\partial t}, \frac{\partial y}{\partial t}, \frac{\partial z}{\partial t}\right)$$

Now

$$\left(\frac{\partial x}{\partial t}, \frac{\partial y}{\partial t}, \frac{\partial z}{\partial t}\right) = \frac{\partial x}{\partial t}\underline{\underline{i}} + \frac{\partial y}{\partial t}\underline{\underline{j}} + \frac{\partial z}{\partial t}\underline{\underline{k}}$$

is a tangent $\underline{\underline{t}}$ to the curve c and the surface

Then

$$\left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right) = \frac{\partial\phi}{\partial x}\underline{\underline{i}} + \frac{\partial\phi}{\partial y}\underline{\underline{j}} + \frac{\partial\phi}{\partial z}\underline{\underline{k}}$$

is a vector perpendicular to (every) tangent $\underline{\underline{t}}$ to the surface S.

In other words, grad ϕ is a vector, depending on x, y, z, which is perpendicular to the surface S at the point $(x, y, z) \in S$.