Recall: We can find the derivative $d\omega$ of a k-form ω using rules (1) - (7) from previous lectures.

$$(3) \quad d(A\,dx + B\,dy + \cdots) =$$

$$(dA)\wedge dx + (dB)\wedge dy + \cdots$$

where A is a form.

<u>Example</u> Calculate $d\omega$ for

$$\omega = x^2\,dy\wedge dz + y^2\,dz\wedge dx + z^2\,dx\wedge dy.$$

<u>Sol^n</u>

$$d\omega = d(x^2\,dy\wedge dz) + d(y^2\,dz\wedge dx) + d(z^2\,dx\wedge dy)$$

$$= d(x^2)\wedge dy\wedge dz + d(y^2)\wedge dz\wedge dx + d(z^2)\wedge dx\wedge dy$$

$$= 2x\,dx \wedge dy \wedge dz + 2y\,dy \wedge dz \wedge dx$$
$$+ 2z\,dz \wedge dx \wedge dy$$

$$= 2x\,dx \wedge dy \wedge dz$$
$$* - 2y\,dy \wedge dx \wedge dz$$
$$- 2z\,dx \wedge dz \wedge dy$$

$$= 2x\,dx \wedge dy \wedge dz$$
$$+ 2y\,dx \wedge dy \wedge dz$$
$$+ 2z\,dx \wedge dy \wedge dz$$

$$= 2(x+y+z)\,dx \wedge dy \wedge dz.$$

Proposition For any k-form ω we have

$$d(d\omega) = 0$$

Exercise For 0-forms v, ω we have

$$d(v\omega) = (dv)\omega + v(d\omega)$$

Proof $d(v\omega)$

$$= \frac{\partial}{\partial x}(v\omega)\,dx + \frac{\partial}{\partial y}(v\omega)\,dy + \cdots$$

$$= \left(\frac{\partial v}{\partial x}\omega + v\frac{\partial \omega}{\partial x}\right)dx + \left(\frac{\partial v}{\partial y}\omega + v\frac{\partial \omega}{\partial y}\right)dy + \cdots$$

$$= \left(\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy \cdots\right)\omega + v\left(\frac{\partial \omega}{\partial x}dx + \frac{\partial \omega}{\partial y}dy + \cdots\right)$$

$$= (dv)\omega + v(d\omega).$$

Exercise: For 1-forms υ, ω we have

$$d(\upsilon \wedge \omega) = (d\upsilon) \wedge \omega - \upsilon (d\omega)$$

for you to try!

Theorem For a k-form υ and n-form ω we have

$$d(\upsilon \wedge \omega) = (d\upsilon) \wedge \omega + (-1)^{kn} \upsilon (d\omega).$$

Example show that the area of the region S bounded by a simple closed curve $C = \partial S$ in the xy-plane

$$= \frac{1}{2}\int_S dx\wedge dy - dy\wedge dx$$

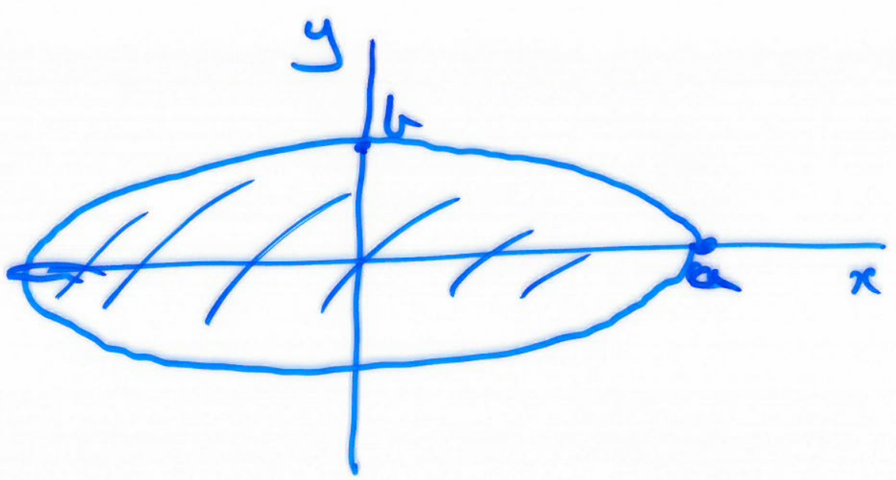
$$= \int_S dx\wedge dy$$

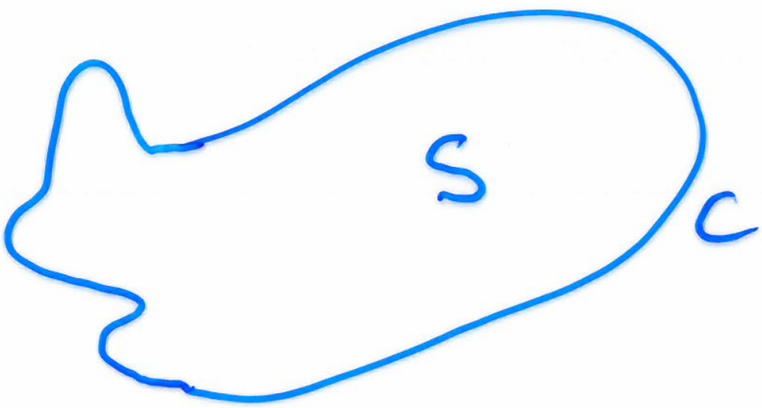
$=$ area of S by definition. [up to sign]

Example Find the area of the region S bounded by the ellipse

$$x = a\cos\theta, \quad y = b\sin\theta$$

Soln





is

$$\frac{1}{2}\int_C x\,dy - y\,dx$$

Sol^n

$$\frac{1}{2}\int_{\partial S} x\,dy - y\,dx$$

$$= \frac{1}{2}\int_S d(x\,dy - y\,dx)$$

Stokes' formula

$$= \frac{1}{2}\int_S d(x\,dy) - d(y\,dx)$$

From above,

required area =

$$\frac{1}{2}\int_{\partial S} x\,dy - y\,dx$$

$$= \frac{1}{2}\int_0^{2\pi} (a\cos\theta)(b\cos\theta)\,d\theta - (b\sin\theta)(-a\sin\theta)\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} ab\,(\cos^2\theta + \sin^2\theta)\,d\theta$$

$$= \frac{1}{2}ab\int_0^{2\pi} d\theta$$

$$= \pi ab.$$