Proposition Let $\omega = F(x,y,z)$ be a 0-form. Suppose that

$$F_{xy} = F_{yx}, \quad F_{xz}\ F_{zx}, \quad F_{yz} = F_{zy}.$$

(Here: $F_x = \frac{\partial F}{\partial x}$, $F_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right), \dots$)

Then

$$d(d\omega) = 0.$$

Proof

$$d(d\omega) = d\left(F_x\,dx + F_y\,dy + F_z\,dz\right)$$

$$= \left(F_{xx}\,dx + F_{xy}\,dy + F_{xz}\,dz\right)\wedge dx$$

$$+ \left(F_{yx}\,dx + F_{yy}\,dy + F_{yz}\,dz\right)\wedge dy$$

$$+ \left(F_{zx}\,dx + F_{zy}\,dy + F_{zz}\,dz\right)\wedge dz$$

$$= F_{xx}\,\cancelto{0}{dx\wedge dx} + F_{xy}\,\cancel{dy}\wedge dx + F_{xz}\,dz\wedge dx$$

$$+ F_{yx}\,\cancel{dx}\wedge dy + F_{yy}\,dy\wedge dy + \dots$$

etc.

$= 0$.

<u>Example</u> Prove that

$$\omega = (3x^2 - 6yz)\,dx + (2y + 3xz)\,dy + (1 - 4xyz^2)\,dz$$

does <u>not</u> arise as $\omega = d\nu$ for any 0-form ν.

<u>soln</u> It suffices to show that

$d\omega \neq 0$

$$d\omega = (6x\,dx - 6z\,dy - 6y\,dz)\wedge dx + (3z\,dx + 2\,dy + 3x\,dz)\wedge dy + (-4yz^2\,dx - 4xz^2\,dy - 8xyz\,dz)\wedge dz$$

$$= 6x\,dx\wedge dx - 6z\,dy\wedge dx - 6y\,dz\wedge dx$$

$+ \cdots\cdots$ etc

$= \dots$

$\neq 0.$ Done

Differentiation of k-forms

A 2-form is an expression such as

$$\omega = A\,dx\wedge dy + B\,dy\wedge dz + C\,dz\wedge dx + \cdots$$

where A, B, C are functions of $x, y, z, \dots$.

A 3-form is an expression such as

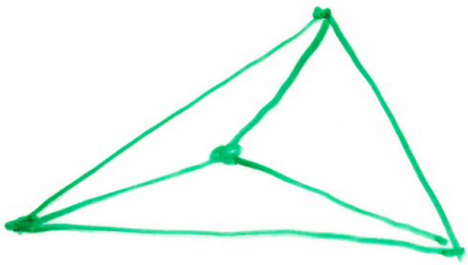
$$\omega = A\,dx\wedge dy\wedge dz + B\,dx\wedge dy\wedge dt + \cdots$$

where $A, B \dots$ are functions of $x, y, z, t, \dots$.

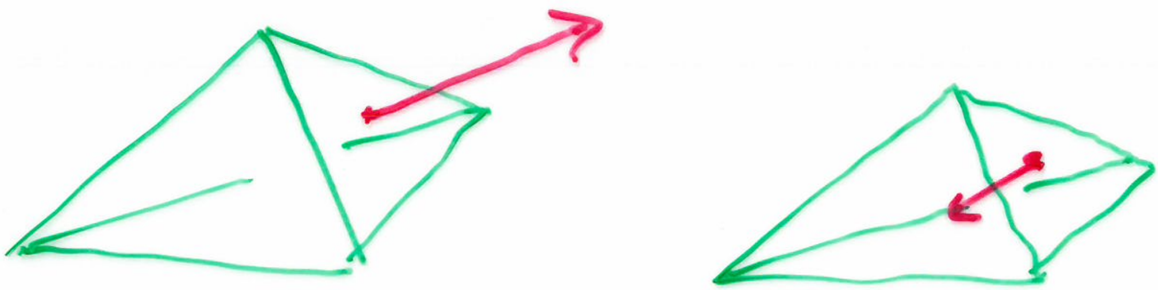
To understand integrals of 2-forms we need to understand how to

integrate a constant 2-form over oriented planar triangles.
(area)

For integrals of 3-forms we need to understand how to integrate constant 3-forms over tetrahedra:



An orientation of such a tetrahedron can be specified by an arrow on its surface pointing either inwards or outwards:



Given a 2-form ω we can define a 3-form $d\omega$ such that

$$\int_{\partial S} \omega = \int_S d\omega$$

where S is an oriented 3-dimensional region.

The derivative $d\omega$ satisfies rules 1-6 from last week, and also:

7. $(dx \wedge dy) \wedge dz = dx \wedge (dy \wedge dz)$

We usually write

$dx \wedge dy \wedge dz$.

<u>Exercise</u> Calculate $d\omega$ for

$\omega = x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy$