Differentiation of 1-forms

For a 1-form ω we have a 2-form $d\omega$. The definition of $d\omega$ is crafted to ensure Stokes' formula

$$\int_{\partial S} \omega = \int_S d\omega$$

holds under reasonable hypotheses.

The following computational rules hold:

for 1-forms ω and ω'

and for 0-forms $A, B, C, \dots$

in variables $x, y, z, \dots$

1. $d(\omega + \omega') = d\omega + d\omega'$

2. $dA = \frac{\partial A}{\partial x} dx + \frac{\partial A}{\partial y} dy + \frac{\partial A}{\partial z} dz + \dots$

3. $d(A\,dx + B\,dy + \dots) =$

$$(dA) \wedge dx + (dB) \wedge dy + \dots$$

4. $dx \wedge dx = 0, \quad dy \wedge dy = 0, \dots$

5. $dx \wedge dy = -dy \wedge dx, \quad dx \wedge dz = -dz \wedge dx, \dots$

6. $(\omega + \omega') \wedge dx = \omega \wedge dx + \omega' \wedge dx, \dots$

<u>Example</u> Calculate $d\omega$ for

$$\omega = xy\,dz + yz\,dx + zx\,dy$$

<u>Sol^n</u>

$$\begin{aligned}
d\omega &= d(xy\,dz) + d(yz\,dx) + d(zx\,dy) \\
&= d(xy)\wedge dz + d(yz)\wedge dx + d(zx)\wedge dy \\
&= (y\,dx + x\,dy)\wedge dz \\
&\quad + (z\,dy + y\,dz)\wedge dx \\
&\quad + (z\,dx + x\,dz)\wedge dy \\
&= y\,dx\wedge dz + x\,dy\wedge dz \\
&\quad + z\,dy\wedge dx + y\,dz\wedge dx \\
&\quad + z\,dx\wedge dy + x\,dz\wedge dy \\
&= -y\,\cancel{dz\wedge dx} + x\,\cancel{dy\wedge dz} \\
&\quad - z\,\cancel{dx\wedge dy} + y\,\cancel{dz\wedge dx} \\
&\quad + z\,\cancel{dx\wedge dy} - x\,\cancel{dy\wedge dz} \\
&= 0.
\end{aligned}$$

Last lecture we saw that the above six rules of differentiation were enough to ensure:

for $\omega = A\,dx + B\,dy$

we have

$$\boxed{d\omega = \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right) dx \wedge dy} \quad . \quad (*)$$

To ~~justify~~ motivate the rules we'll explain why $(*)$ is what is needed for Stokes' formula to hold.

So suppose

$$\omega = A\,dx + B\,dy$$

where A, B are functions of x, y.

We want to <u>define</u>

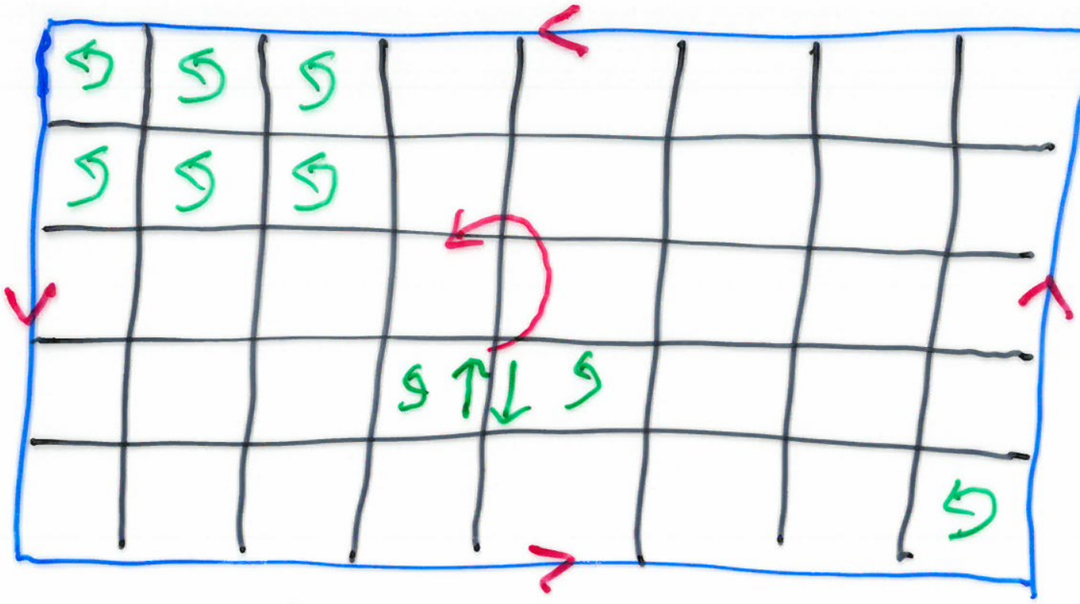
$$d\omega = C\,dx \wedge dy$$

such that

$$\boxed{\int_{\partial S} A\,dx + B\,dy = \int_S C\,dx \wedge dy}$$

What does C have to be?

For simplicity let's suppose S is an oriented region in the xy-plane, with boundary ∂S oriented accordingly.



$S = S_1 \cup S_2 \cup \ldots\ldots \cup S_n$

Note that

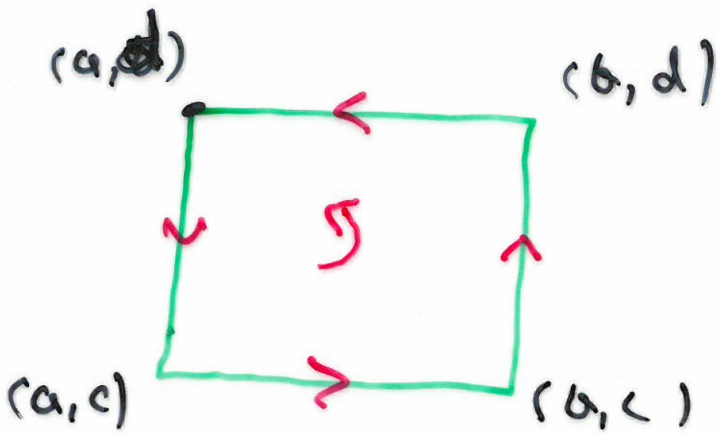
$$\int_{\partial S} A\,dx + B\,dy = \sum_{i=1}^{n} \int_{\partial S_i} A\,dx + B\,dy$$

so, for each small square S_i

we just need

$$\boxed{\int_{\partial S_i} A\,dx + B\,dy = \int_{S_i} c\,dx \wedge dy}$$

Suppose S_i is the square



$$a \leq x \leq b$$

$$c \leq y \leq d.$$

Assume a good $c\, dx \wedge dy$ exists.

we have

$$\int_{\partial S_i} A\,dx + B\,dy$$

$$= \int_a^b A(x,c)\,dx + \int_c^d B(b,y)\,dy$$

$$+ \int_b^a A(x,d)\,dx + \int_d^c B(a,y)\,dy$$

$$= \int_c^d \big(B(b,y) - B(a,y) \big)\, dy$$

$$- \int_a^b \big(A(x,d) - A(x,c) \big)\, dx$$

$$\overset{\text{FTC}}{=} \int_c^d \left(\int_a^b \frac{\partial B}{\partial x}\, dx \right) dy - \int_a^b \left(\int_c^d \frac{\partial A}{\partial y}\, dy \right) dx$$

$$= \int_{S_i} \frac{\partial B}{\partial x}\, dx \wedge dy - \int_{S_i} \frac{\partial A}{\partial y}\, dx \wedge dy$$

$$= \int_{S_i} \underbrace{\left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y} \right)}_{C} dx \wedge dy$$

Then, we need

$$\boxed{d\omega = c\, dx \wedge dy = \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right) dx \wedge dy}$$