Example Evaluate

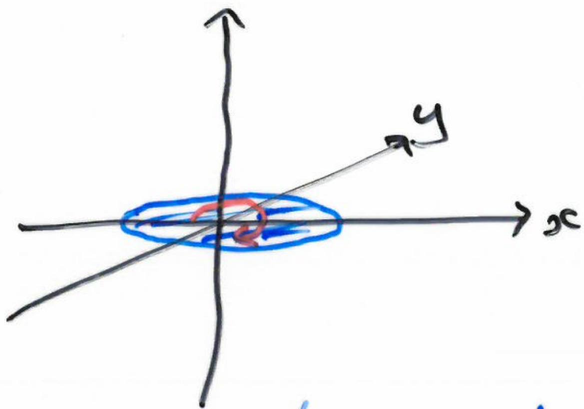
$$I = \int_S 3\, dx \wedge dy + 4\, dy \wedge dz$$

where S is the disk

$$S = \{(x,y,z) \in \mathbb{R}^3 : z=0,\ x^2+y^2 \le 1\}$$

with clockwise orientation.

Sol^n



From the definition of an integral we see that

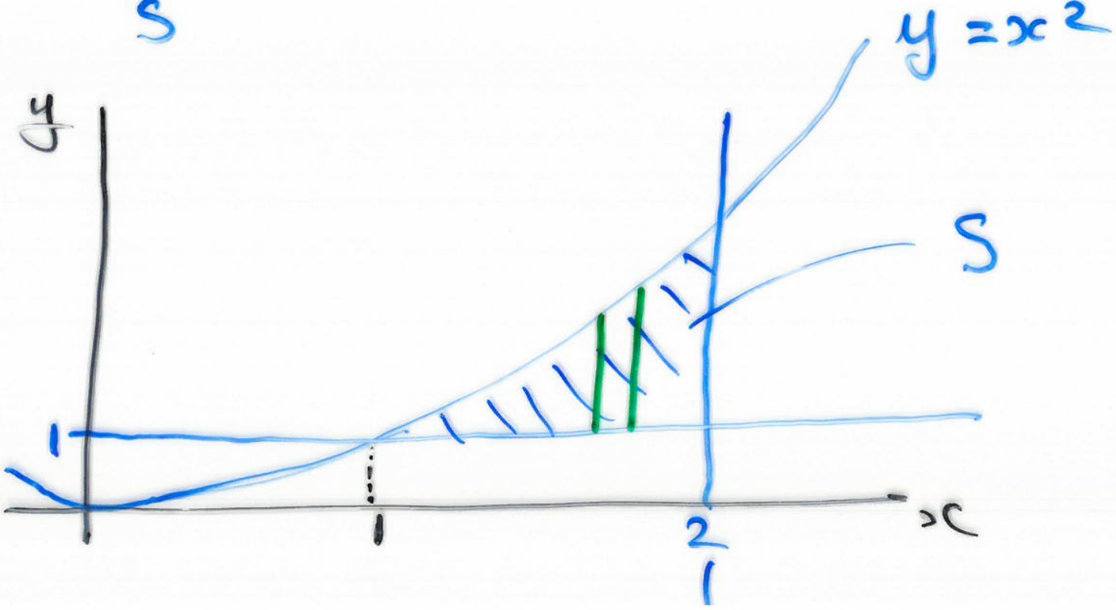
$$I = 3 \times (-1) \text{ area of disk} = -3\pi$$

<u>Example</u> Let S be the region in the xy-plane bounded by $y = x^2$, $x = 2$, $y = 1$. Let S have an anticlockwise orientation. Evaluate

$$I = \int_S (x^2 + y^2 + z^2)\, dx \wedge dy$$

<u>Sol^n</u> On S we have $z = 0$ and thus

$$I = \int_S (x^2 + y^2)\, dx \wedge dy$$



Subdivide S into thin strips parallel to y-axis

We can write

$$I = \int_{x=1}^{x=2} \left(\int_{y=1}^{y=x^2} (x^2+y^2)\, dy \right) dx$$

$$I = \int_{x=1}^{x=2} \left(x^2 y + \frac{y^3}{3} \Big|_{y=1}^{y=x^2} \right) dx$$

$$= \int_{x=1}^{2} \left(x^4 + \tfrac{1}{3}x^6 - x^2 - \tfrac{1}{3} \right) dx$$

$$= \frac{x^5}{5} + \frac{1}{21}x^7 - \frac{1}{3}x^3 - \frac{1}{3}x \Big|_{x=1}^{x=2}$$

$$= \frac{1006}{105}$$

Example Evaluate

$$I = \int_S (x+y+z)\, dx \wedge dy$$

where S is the oriented planar rectangle with vertices $(0,0,1)$, $(0,1,1)$, $(1,1,1)$, $(1,0,1)$ in that order.

Sol^n See 2014-15 online lecture notes.

Differentiation of 1-forms

For a 1-form ω, and a 2-dimensional oriented region S, we'd like to define a 2-form

$$d\omega$$

such that

$$\boxed{\int_{\partial S} \omega = \int_S d\omega \qquad (*)}$$

The definition of $d\omega$ is determined by the desire to have equation $(*)$ hold.

Definition of the derivative of an n-form

For n-forms ω, ω'

$$\boxed{d(\omega + \omega') = d\omega + d\omega'}$$

For functions $A, B, \ldots$ in variables $x, y, z, \ldots$

$$\boxed{dA = \frac{\partial A}{\partial x} dx + \frac{\partial A}{\partial y} dy + \frac{\partial A}{\partial z} dz}$$

$$\boxed{\begin{aligned} &d(A\,dx + B\,dy + \ldots) = \\ &\quad dA \wedge dx + dB \wedge dy + \ldots \end{aligned}}$$

$$\boxed{dx \wedge dx = 0}$$

$$\boxed{dx \wedge dy = -dy \wedge dx}$$

Example find $d\omega$

where

$$\omega = A\,dx + B\,dy$$

where A, B are functions of x, y.

Sol^n

$$d\omega = d(A\,dx + B\,dy)$$

$$= d(A\,dx) + d(B\,dy)$$

$$= dA \wedge dx + dB \wedge dy$$

$$= \left(\frac{\partial A}{\partial x}dx + \frac{\partial A}{\partial y}dy\right) \wedge dx + \left(\frac{\partial B}{\partial x}dx + \frac{\partial B}{\partial y}dy\right) \wedge dy$$

$$= \frac{\partial A}{\partial x}\underbrace{\cancel{dx \wedge dx}}_{=0} + \frac{\partial A}{\partial y}dy \wedge dx + \frac{\partial B}{\partial x}dx \wedge dy + \frac{\partial B}{\partial y}\underbrace{\cancel{dy \wedge dy}}_{=0}$$

$$= \frac{\partial A}{\partial y} dy \wedge dx + \frac{\partial B}{\partial x} dx \wedge dy$$

$$= -\frac{\partial A}{\partial y} dx \wedge dy + \frac{\partial B}{\partial x} dx \wedge dy$$

$$= \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y} \right) dx \wedge dy$$