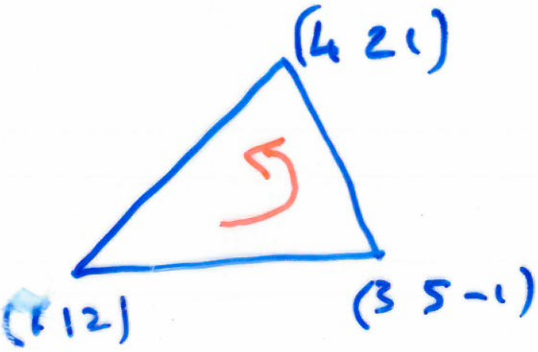
Example Evaluate

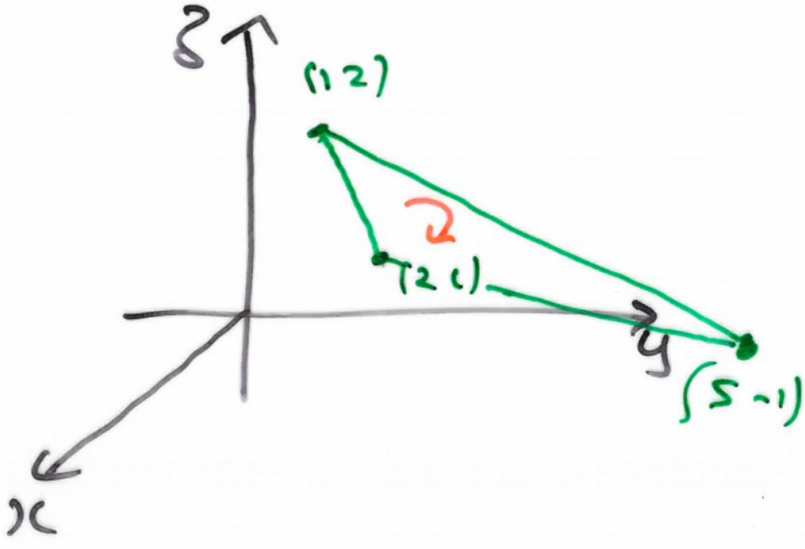
$$I = \int_S dy\wedge dz + dz\wedge dx + dx\wedge dy$$

where S is the oriented triangle with vertices $(1,1,2)$, $(3,5,-1)$, $(4,2,1)$ in that order.

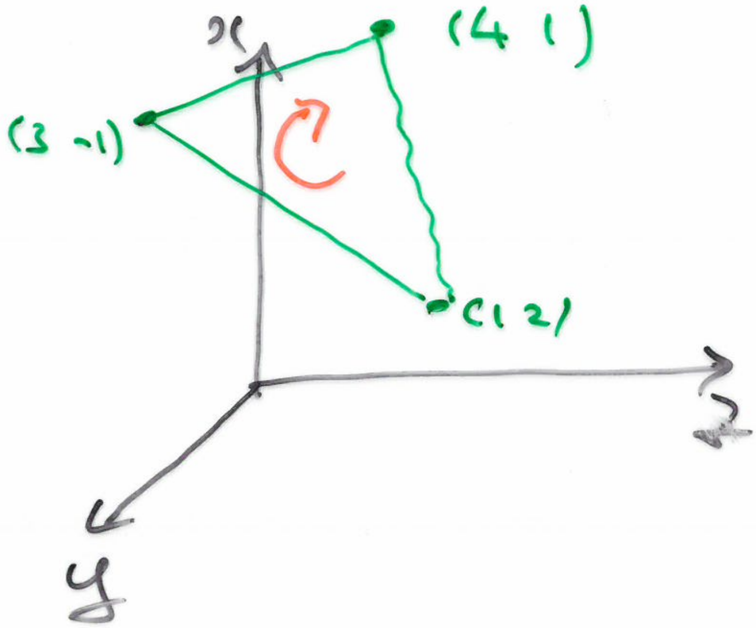
Sol^n

$$I = \int_S dy\wedge dz + \int_S dz\wedge dx + \int_S dx\wedge dy$$

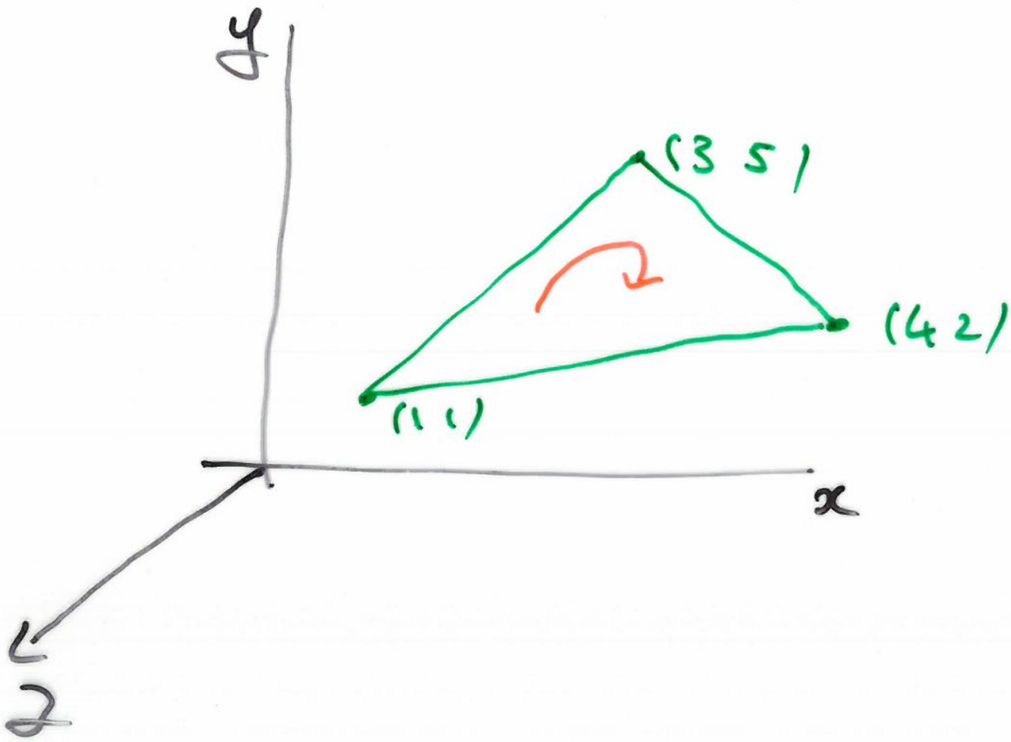




Area of triangle $= -\frac{1}{2}\begin{vmatrix} -1 & 3 \\ 1 & -2 \end{vmatrix} = \frac{1}{2}$



Area of triangle $= \frac{1}{2}\begin{vmatrix} 2 & 3 \\ -3 & -1 \end{vmatrix} = \frac{7}{2}$



Area $= -\frac{1}{2}\begin{vmatrix} 2 & 3 \\ 4 & 1 \end{vmatrix} = 5$

$$\int_S dy \wedge dz = -\frac{1}{2}$$

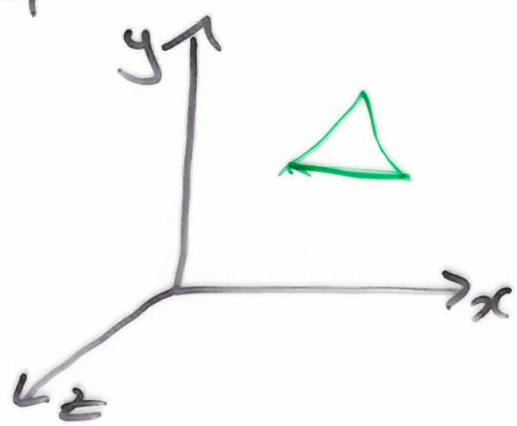
$$\int_S dz \wedge dx = -\frac{7}{2}$$

$$\int_S dx \wedge dy = -5$$

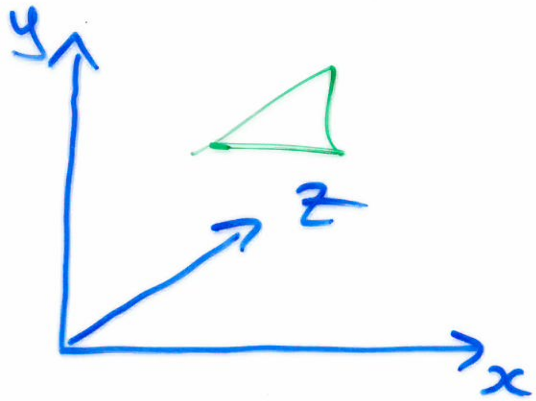
so

$$I = -\frac{1}{2} - \frac{7}{2} - 5 = -9$$

<u>Remark</u> $\int_S dx \wedge dy$ refers to the orientation



$\int_S dy \wedge dx$ refers to the orientation



So

$$\boxed{\int_S dx \wedge dy = - \int_S dy \wedge dx}$$

we write

$$\boxed{dx \wedge dy = - dy \wedge dx}$$

<u>Conventions</u> $dx \wedge dy$, $dy \wedge dz$, $dz \wedge dx$ are the "orientations" when we view things from the positive third axis.

So

$$\int_S dy \wedge dz = - \int_S dz \wedge dy$$

etc.

and

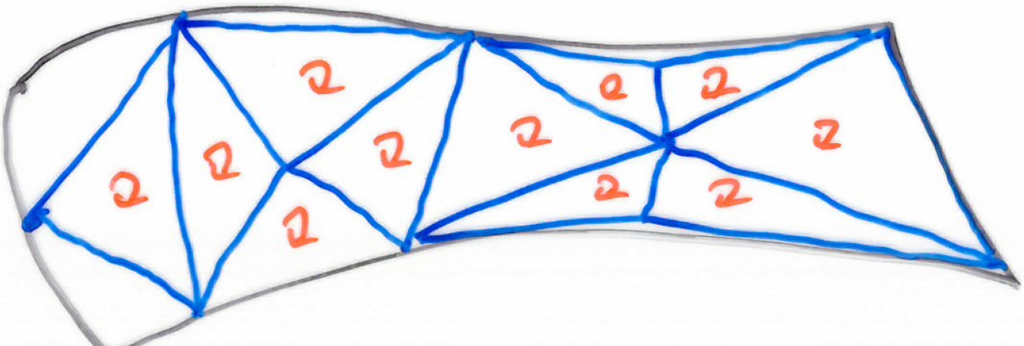
$$dy \wedge dz = - dz \wedge dy$$

etc.

Integration of 2-forms

Let S denote a 2-dimensional region in $\mathbb{R}^3$ with some choice of orientation

We can approximate S by a union of oriented planar triangles



$P = T_1 \cup T_2 \cup \cdots \cup T_k$
union of k oriented triangles.

Suppose we have a sequence of approximations $P_1, P_2, P_3, \ldots$

where:

1) The approximation P_i gets better as $i \to \infty$

2) the area of the largest triangle in P_i tends to 0 as $i \to \infty$

We define

$$\int_S A(x,y,z)\,dx\wedge dy + B(x,y,z)\,dy\wedge dz + C(x,y,z)\,dz\wedge dx$$

$$= \lim_{i\to\infty} \sum_{T_j \text{ in } P_i} \int_{T_j} A(x_j,y_j,z_j)\,dx\wedge dy + B(x_j,y_j,z_j)\,dy\wedge dz + C(x_j,y_j,z_j)\,dz\wedge dx$$

where (x_j,y_j,z_j) lies in T_j