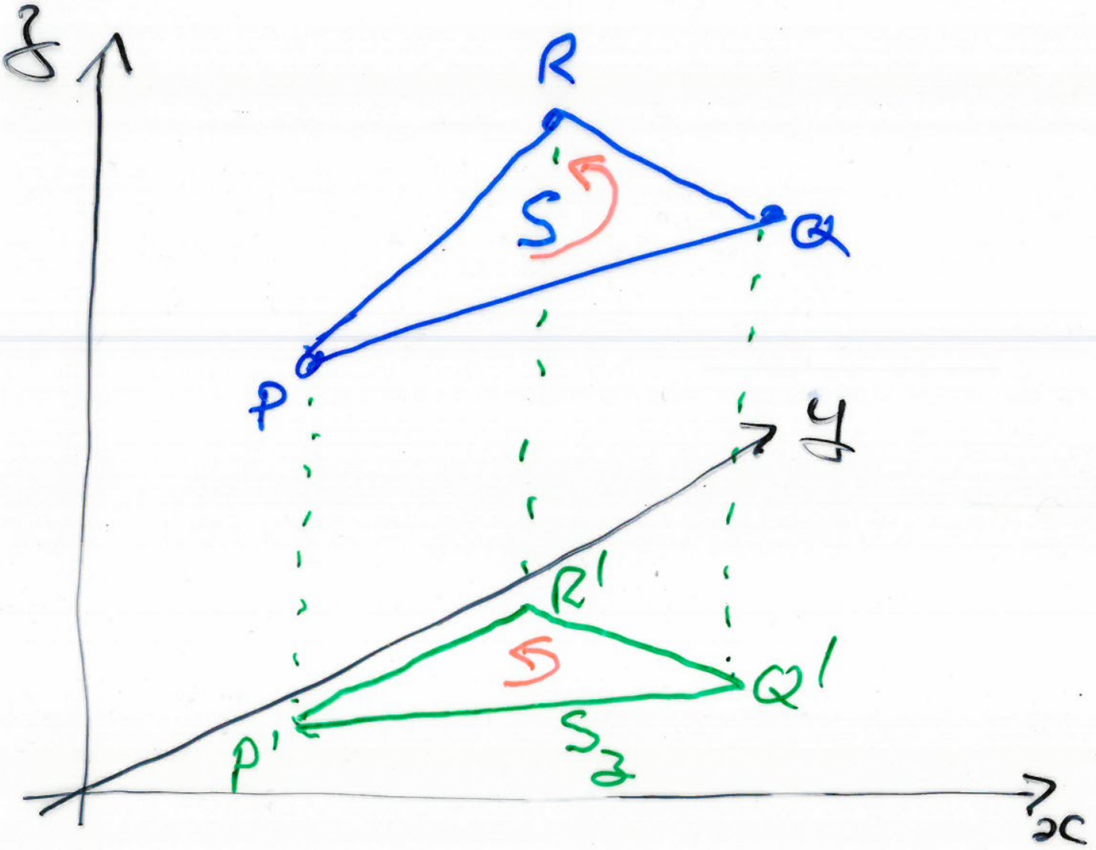
Constant 2-forms

Let S denote an oriented triangle in $\mathbb{R}^3$.



Let S_z denote the image of the triangle S in the xy-plane under the projection

$$p_z : \mathbb{R}^3 \longrightarrow \mathbb{R}^2, \quad (x,y,z) \longmapsto (x,y)$$

for any constant $A \in \mathbb{R}^3$

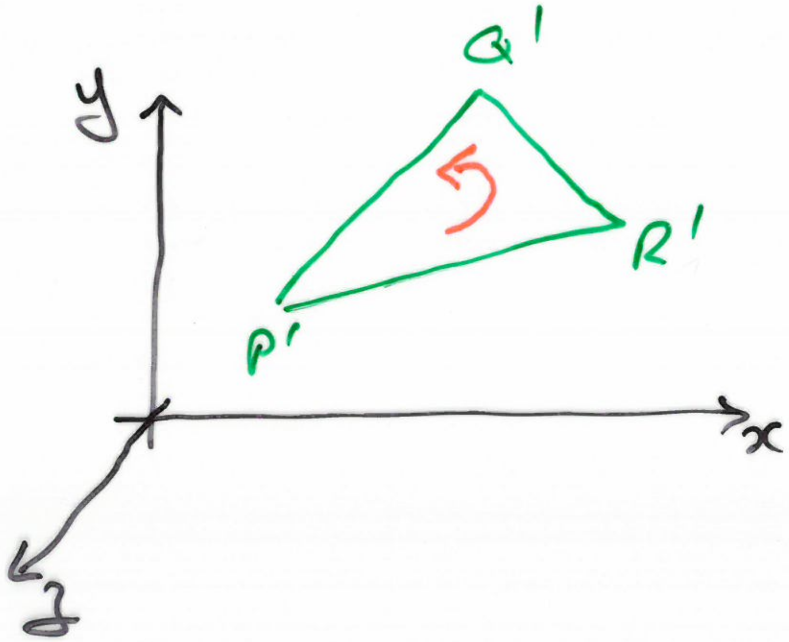
let

$$\int_S A \, dx \wedge dy$$

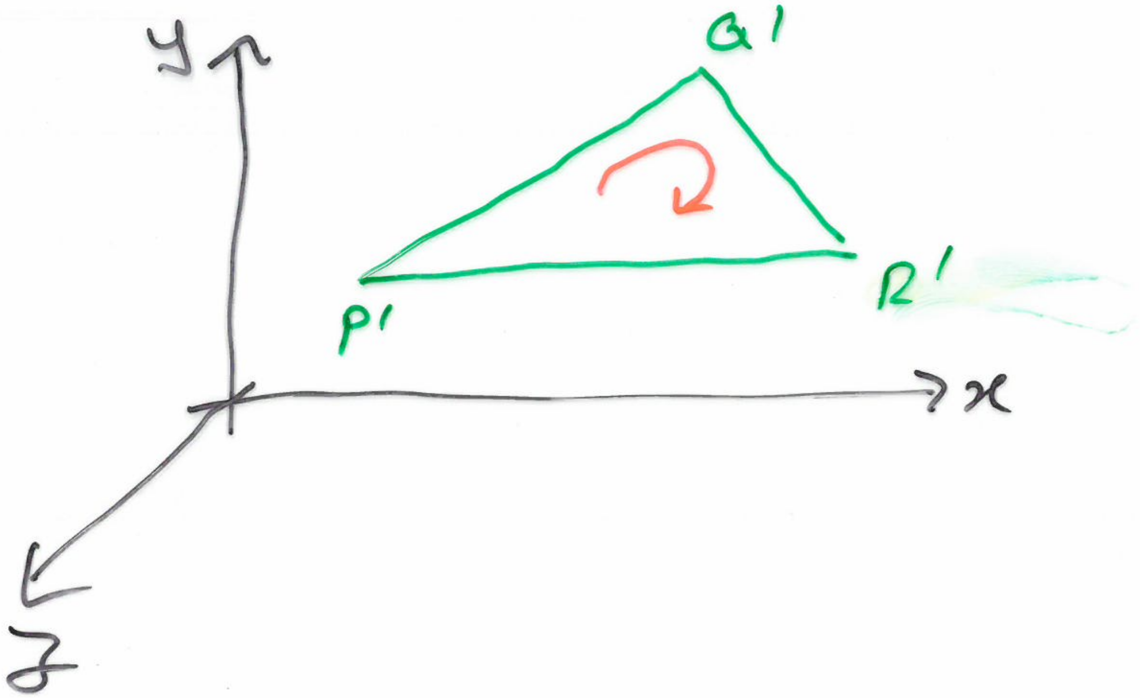
denote

$$\pm A \times (\text{area of } S_z)$$

with sign $+1$ if



and sign -1 if



Similarly define for $B, C \in \mathbb{R}$

$$\int_S C \, dz \wedge dx \quad \text{and} \quad \int_S B \, dy \wedge dz$$

<u>Defn</u>

$$\int_S A \, dx \wedge dy + B \, dy \wedge dz + C \, dz \wedge dx$$

$$= \int_S A \, dx \wedge dy + \int_S B \, dy \wedge dz + \int_S C \, dz \wedge dx$$

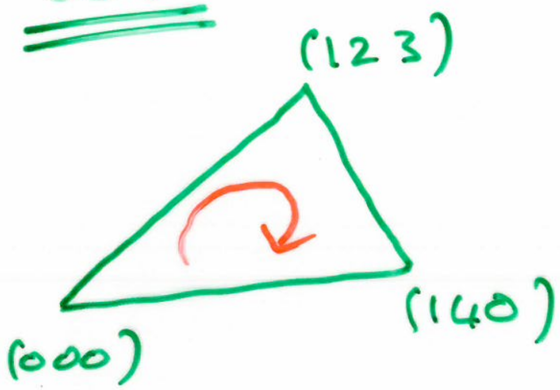
Example

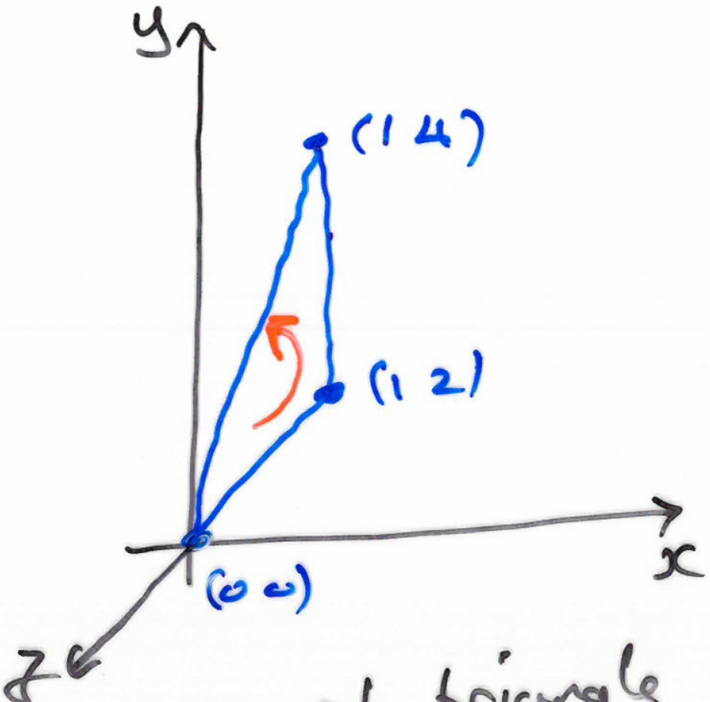
Evaluate

$$I = \int_S dx \wedge dy + 3\, dz \wedge dx$$

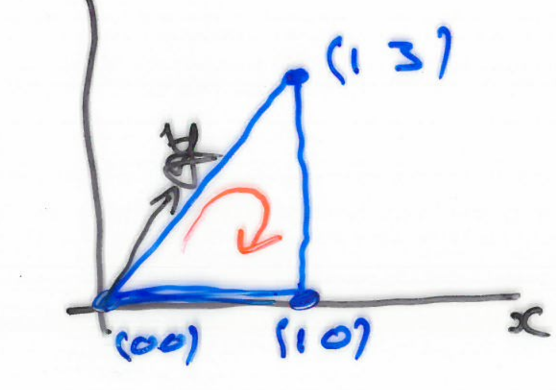
over the oriented triangle S with vertices $(0,0,0)$, $(1,2,3)$, $(1,4,0)$ in that order.

Sol^n





area of triangle $= 1$



$$\text{Area} = \frac{1}{2}3 = \frac{3}{2}$$

So

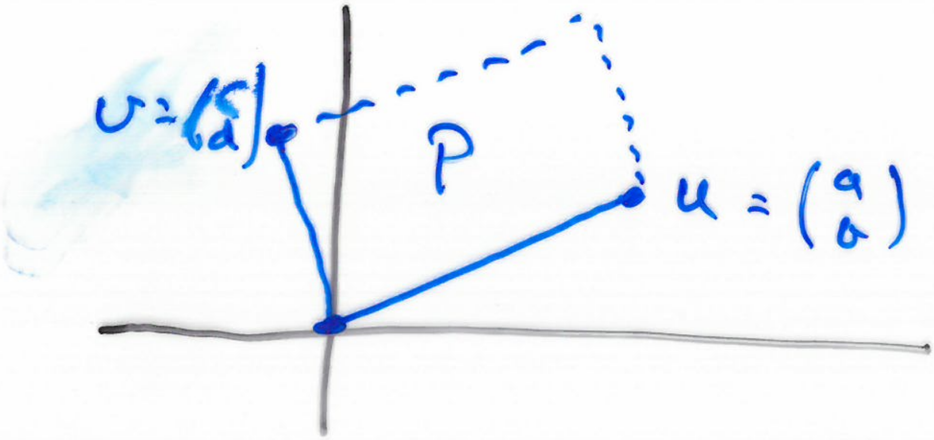
$$I = \int_S dx\wedge dy + 3\, dz \wedge dx$$

$$= \int_S dx\wedge dy + \int_S 3\, dz\wedge dx$$

$$= +1 + -3\cdot\frac{3}{2}$$

$$= 1-\frac{9}{2} = -\frac{7}{2}.$$

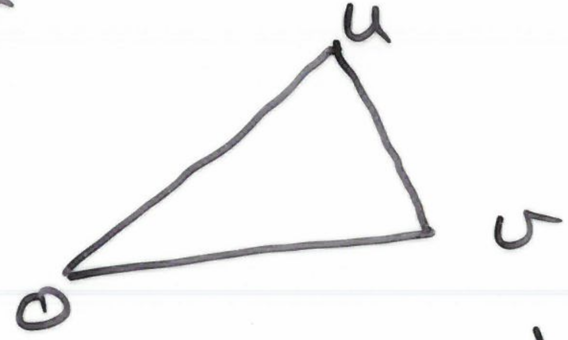
Recall: area of parallelogram



Prop^n Area of $P = \pm \begin{vmatrix} a & c \\ b & d \end{vmatrix} = \pm(ad-bc)$.

Proof See MA 180

Corollary Area of the triangle



is $\pm \frac{1}{2} \det \begin{pmatrix} | & | \\ u & v \\ | & | \end{pmatrix}$.

Example

Evaluate

$$I = \int_S dy \wedge dz + dz \wedge dx + dx \wedge dy$$

where S is the oriented triangle with vertices $(1,1,2)$, $(3,5,-1)$, $(4,2,1)$ in that order.