MA286: Tutorial Problems 2014-15

Tutorials: Tuesday, 6-7pm, Venue = IT202
Thursday, 2-3pm, Venue = IT207

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For those questions taken from the Schaum Outline Series book *Advanced Calculus* by M. Spiegel the question number in the book is given. The book provides worked solutions for many of these questions.

PROBLEMS

1 0-forms on 1-dimensional space

1. Give an interval $S = [a, b] \subset \mathbb{R}$ on which
$$\omega = |x - 4|$$
is a differential 0-form. Then give an interval $S' = [a', b'] \subset \mathbb{R}$ on which ω is not a differential 0-form.

2. Evaluate the integral
$$\int_{\partial S} 2x^2 + x$$
of the differential 0-form $\omega = 2x^2 + x$ over the boundary of the oriented interval $S = [3, -1]$.

3. Evaluate the integral
$$\int_{\partial S} x^3$$
of the differential 0-form $\omega = x^3$ over the boundary of $S = [2, 1] \cup [4, 3] \cup [-2, -1]$.

4. Is
$$\omega = \begin{cases} x\ \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$
a differential 0-form on the interval $S = [-1, 1]$? [See 4.4(b)]

2 1-forms on 1-dimensional space

1. Find a differential 0-form ω whose derivative $d\omega$ is the differential 1-form
$$d\omega = (x^2 + 2x)\,dx\ .$$

2. Find a differential 0-form ω whose derivative $d\omega$ is the differential 1-form
$$d\omega = (x+2)\sin(x^2 + 4x - 6)\,dx\ .$$
[See 5.14(a)]

3. Find a differential 0-form ω whose derivative $d\omega$ is the differential 1-form
$$d\omega = \frac{6-x}{(x-3)(2x+5)}\,dx\ .$$
[See 5.20]

4. Find a differential 0-form ω whose derivative $d\omega$ is the differential 1-form
$$d\omega = \frac{1}{5+3\cos x}\,dx\ .$$
[See 5.21]

3 Fundamental theorem of calculus

1. Evaluate the integral
$$\int_S \frac{1}{\sqrt{(x+2)(3-x)}}\,dx$$
of the differential 1-form $\omega = dx/\sqrt{(x+2)(3-x)}$ over the oriented interval $S = [-1, 1]$. [See 5.14(c)]

2. Evaluate the integral
$$\int_S \frac{1}{(x^2-2x+4)^{3/2}}\,dx$$
of the differential 1-form $\omega = (x^2-2x+4)^{-3/2}\,dx$ over the oriented interval $S = [2, 1]$. [See 5.15]

3. Evaluate the integral
$$\int_S \frac{1}{x(\ln x)^3}\,dx$$
of the differential 1-form $\omega = dx/x(\ln x)^3$ over the oriented interval $S = [e, e^2]$. [See 5.16]

4. Give an informal proof of Stokes' formula $\int_{\partial S} \omega = \int_S d\omega$ for $S = [a, b] \subset \mathbb{R}$ and $\omega = f(x)\colon \mathbb{R} \to \mathbb{R}$ a differentiable function.

4 0-forms on n-dimensional space

1. Let S denote the oriented line segment in the plane going from the point $A = (1, 2)$ to the point $B = (-2, 3)$. Evaluate the integral

$$\int_{\partial S} x^2 + xy + y^2$$

of the differential 0-form $\omega = x^2 + xy + y^2$ over the boundary of S.

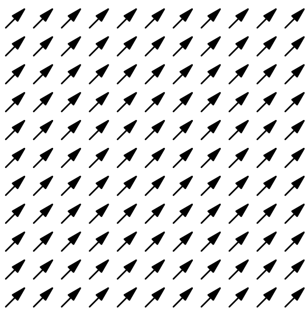
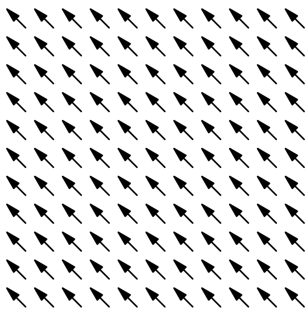
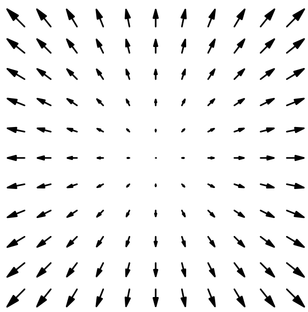
2. Let S denote the oriented line segment on the z-axis in $\mathbb{R}^3$ going from $z = 1$ to $z = 2$. Evaluate the integral

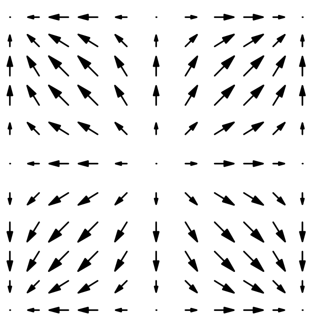
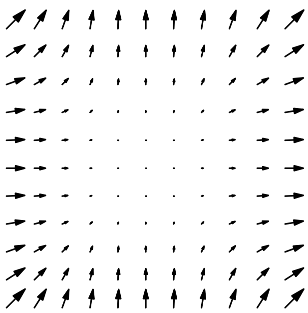
$$\int_{\partial S} z\, e^{x^2+y^2}$$

of the differential 0-form $\omega = z\, e^{x^2+y^2}$ over the boundary of S.

5 1-forms on n-dimensional space

1. Match the five pictures of flows

(a) 
(b) 
(c) 

(d) 
(e) 

to the five differential 1-forms: (i) $\omega = x^2\, dx + y^2\, dy$, (ii) $\omega = \sin(\pi x)\, dx + \sin(\pi y)\, dy$, (iii) $\omega = x\, dx + y\, dy$, (iv) $\omega = dx + dy$, (v) $\omega = -dx + dy$.

2. In a constant force field the displacement of a particle

- from $(0, 0, 0)$ to $(4, 0, 0)$ needs 3 units of work;

- from $(1,-1,0)$ to $(1,1,0)$ needs 2 units of work;
- from $(0,0,0)$ to $(3,0,2)$ needs 5 units of work.

Determine the differential 1-form that describes "work".

6 Integration of constant 1-forms

1. Evaluate the integral
$$\int_S 2\,dx + 3\,dy + 5\,dz$$
of the differential 1-form $\omega = 2\,dx + 3\,dy + 5\,dz$ on the line segment S in $\mathbb{R}^3$ starting at point $P = (3,12,4)$ and ending at point $Q = (11,14,-7)$.

2. If work is given by the 1-form $3\,dx + 4\,dy - dz$ find all points which can be reached from the origin $(0,0,0)$ without work. Describe the set of these points geometrically.

7 Integration of 1-forms

1. Evaluate the integral
$$\int_S (x^2 - y)\,dx + (y^2 + x)\,dy$$
of the differential 1-form $\omega = (x^2 - y)\,dx + (y^2 + x)\,dy$ where $S \subset \mathbb{R}^2$ is the segment of the parabola $x = t$, $y = t^2 + 1$ from the point $(0,1)$ to the point $(1,2)$. [See 10.1]

2. Evaluate the integral
$$\int_C \omega$$
of the 1-form
$$\omega = (3x^2 - 6yz)\,dx + (2y + 3xz)\,dy + (1 - 4xyz^2)\,dz$$
where C is the straight line from $(0,0,0)$ to $(1,1,1)$. [See 10.2(c)]

3. Evaluate the integral
$$\int_C \omega$$
of the 1-form
$$\omega = (3x^2 - 6yz)\,dx + (2y + 3xz)\,dy + (1 - 4xyz^2)\,dz$$
where C is the curve $x = t$, $y = t^2$, $z = t^3$ from $(0,0,0)$ to $(1,1,1)$. [See 10.2(c)]

4. Evaluate the integral
$$\int_{\partial S} (2xy - x^2)\, dx + (x + y^2)\, dy$$
where ∂S is the boundary of the region S bounded by the two curves $y = x^2$ and $y^2 = x$. Assume an anti-clockwise orientation on ∂S. [See 10.6]

8 Differentiation of 0-forms

1. Determine the 1-form $d\omega$ arising as the derivative of the 0-form $\omega = x^2 e^{y/x}$. [See 6.16(a)]

2. Find a 0-form ω whose derivative is
$$d\omega = (3x^2 y - 2y^2)\, dx + (x^3 - 4xy + 6y^2)\, dy.$$
[See 6.16(b)]

9 Partial derivatives

1. Suppose $U = z \sin(y/x)$ where $x = 3r^2 + 2s$, $y = 4r - 2s^3$ and $z = 2r^2 - 3^2$. Calculate $\partial u/\partial r$ and $\partial U/\partial s$. [See 6.22]

10 Fundamental Theorem of Calculus again

1. Evaluate
$$\int_S (6xy^2 - y^3)dx + (6x^2 y - 3xy^2)dy \ .$$
where S is some path from $(1, 2)$ to $(3, 4)$. Explain why the integral is independent of the choice of path from $(1, 2)$ to $(3, 4)$. [See 10.14]

2. Evaluate
$$\int_S (2xy - y^4 + 3)dx + (x^2 - 4xy^3)dy \ .$$
where S is some path from $(1, 0)$ to $(2, 1)$. Explain why the integral is independent of the choice of path from $(1, 0)$ to $(2, 1)$. [See 10.48]

3. Prove that the differential 1-form

$$\omega = (3x^2 - 6yz)\,dx + (2y + 3xz)\,dy + (1 - 4xyz^2)\,dz$$

does not arise as the derivative $\omega = d\nu$ of any 0-form ν on $S - \mathbb{R}^3$. [See Questions 2 and 3 of Section 7 above]

4. Prove Stokes' formula $\int_{\partial S} \omega = \int_S d\omega$ for $\omega = f(x, y) \colon \mathbb{R}^2 \to \mathbb{R}$ a continuously differentiable function and $S \subset \mathbb{R}^2$ an oriented curve with differentiable parametrization $x = g(t)$, $y = h(t)$.

11 Constant 2-forms

1. Evaluate the integral

$$\int_S dx \wedge dy + 3dx \wedge dz$$

of the 2-form $\omega = dx \wedge dy + 3dx \wedge dz$ over the oriented triangle S with vertices $(0, 0, 0)$, $(1, 2, 3)$, $(1, 4, 0)$ in that order.

2. Evaluate the integral

$$\int_S dy \wedge dz + dz \wedge dx + dx \wedge dy$$

of the 2-form $\omega = dy \wedge dz + dz \wedge dx + dx \wedge dy$ over the oriented triangle S with vertices $(1, 1, 1)$, $(3, 5, -1)$, $(4, 2, 1)$ in that order.

3. Evaluate the integral

$$\int_S 3dx \wedge dy$$

of the 2-form $\omega = 3dx \wedge dy$ over the region $S = \{(x, y) \in \mathbb{R}^2 \;:\; x^2 + y^2 \leq 1\}$ where S is given a clockwise rotation when viewed from the positive z-axis.

12 More integration of 2-forms

1. Evaluate the integral

$$\int_S 3\,dx \wedge dy + 4\,dy \wedge dz$$

over the region $S = \{(x, y) \in \mathbb{R}^2 \;:\; x^2 + y^2 \leq 1\}$ where S is given a clockwise rotation when viewed from the positive z-axis.

2. Let S be the region in the xy-plane bounded by $y = x^2$, $x = 2$ and $y = 1$. Let S have an anti-clockwise orientation. Evaluate

$$\int_S (x^2 + y^2 + z^2)\, dx \wedge dy.$$

[See 9.1]

3. Let S be the region in the xy-plane bounded by $y = x^2$, $x = 2$ and $y = 1$. Let S have an anti-clockwise orientation. Evaluate

$$\int_S (x^2 + y^2 + z^2)\, dy \wedge dz.$$

4. Let S be the region in the xy-plane bounded by the curves $y = x^2$, $y = \sqrt{2 - x^2}$, $x = 0$ and $x = 1$. Let S have an anti-clockwise orientation. Evaluate

$$\int_S xy\, dx \wedge dy.$$

[See 9.3(b)]

5. Find the volume of the region in $\mathbb{R}^3$ common to the intersecting cylinders $x^2 + y^2 = 4$ and $x^2 + z^2 = 4$. [See 9.4]

13 Differentiation of k-forms

1. Find $d\omega$ for the following forms.

   (a) $\omega = xy\, dz + yz\, dx + zx\, dy$

   (b) $\omega = x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy$

   (c) $\omega = e^{xyz}$

   (d) $\omega = (\cos x)\, dy + (\sin x)\, dz$

   (e) $\omega = (x + y)^2\, dy + (x + y)^2\, dz$

   (f) $\omega = \log x$

   (g) $\omega = x^2$

   (h) $\omega = \sin x$

   (i) $\omega = x$

2. Let $\omega = F(x, y, z)$ be a 0-form and assume $F_{xy} = F_{yx}$, $F_{xz} = F_{zx}$, $F_{yz} = F_{zy}$. Prove that $d(d\omega) = 0$.

3. Let $\omega = F(x, y, z)\, dx + G(x, y, z)\, dy + H(x, y, z)\, dz$ and assume that each of F, G, H satisfy the hypothesis of the preceding question. Prove that $d(d\omega) = 0$.

4. Use the preceding problem to prove that the differential 1-form

$$\omega = (3x^2 - 6yz)\,dx + (2y + 3xz)\,dy + (1 - 4xyz^2)\,dz$$

does not arise as the derivative $\omega = d\nu$ of any 0-form ν on $S - \mathbb{R}^3$.

5. For two differential 0-forms ν, ω prove that

$$d(\nu\omega) = (d\nu)\omega + \nu(d\omega).$$

6. For two differential 1-forms $\nu = A\,dx + B\,dy$, $\omega = C\,dx + D\,dy$ prove that

$$d(\nu \wedge \omega) = (d\nu) \wedge \omega - \nu \wedge (d\omega).$$

14 Stokes' Formula

1. Verify Stokes' Formula $\int_{\partial S} \omega = \int_S d\omega$ for $\omega = (2xy - x^2)\,dx + (x + y^2)\,dy$ and S the region in the xy-plane bounded by $y = x^2$ and $x^2 = y$. [See 10.6]

2. Verify Stokes' Formula $\int_{\partial S} \omega = \int_S d\omega$ for $\omega = (2x - z)\,dy \wedge dz + x^2 y\,dz \wedge dx - xz^2\,dx \wedge dy$ and S the region in $\mathbb{R}^3$ bounded by $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, $z = 1$. (This is a very long and tedious questions to answer!) [See 10.23]

3. Let S denote the region bounded by some ellipse (or other some other simple closed curve) in the xy-plane. Use Stokes' Formula to show that the area of S is given by $\int_{\partial S} x\,dy - y\,dx$. [See 10.8]

4. Calculate the area bounded by the ellipse $x = a\cos\theta$, $y = b\sin\theta$.

5. By considering an oriented 2-dimensional rectangle S in the xy-plane, explain how Stokes' formula $\int_{\partial S} \omega = \int_S d\omega$ leads to the definition of the derivative $d\omega$ of a differential 1-form $\omega = A\,dx + B\,dy$.

15 div, grad, curl

1. Consider the 0-form $\omega = (x^2 + y^2)/2$. Calculate the "gradient" 1-form $d\omega$ and sketch the corresponding vector field on $\mathbb{R}^2$.

2. Find a unit normal to the surface $S \subset \mathbb{R}^3$ defined by the equation

$$2x^2 + 4yz - 5z^2 = -10$$

at the pont $(3, -1, 2) \in S$. [See 7.37]

3. Consider the 1-form $\omega = -y\,dx + x\,dy$. Sketch the corresponding vector field. Then compute the "curl" 2-form $d\omega$. What feature of your sketch is captured by $d\omega$?

4. Consider the vector field $F = xz\mathbf{i} - y^2\mathbf{j} + 2x^2y\mathbf{k}$. Define curl$(F)$ in terms of the derivative of a 1-form and then calculate curl(F).

5. Consider the 0-form $\omega = x^2yz^3$ and the vector field $F = xz\mathbf{i} - y^2\mathbf{j} + 2x^2y\mathbf{k}$. Determine grad$(\omega)$, div$(F)$, curl$(F)$, div$(\phi F)$, div$(\omega F)$, curl$(\omega F)$. [See 7.34]