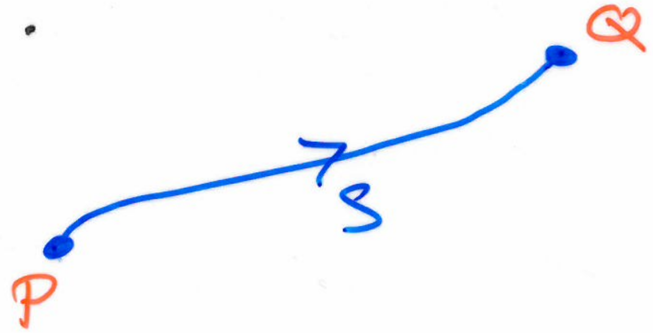
Fundamental Theorem of Calculus again!

Let ω be a 0-form on n-dimensional space.

Let S be a curve in $\mathbb{R}^n$ from P to Q.



Theorem

$$\int_S d\omega = \int_{\partial S} \omega$$

Example Evaluate

$$I = \int_S (y^3+2x)\,dx + 3xy^2\,dy$$

where S is the straight line segment from $P=(0,0)$ to $Q=(1,2)$.

Sol^n (using above theorem)

Consider

$$\omega = xy^3 + x^2$$

Then

$$d\omega = (y^3+2x)dx + (3xy^2)dy$$

So

$$I = \omega\Big|_{(1,2)} - \omega\Big|_{(0,0)} = 9$$

Sol^n (using defn of an integral)

The points $(x=t, y=2t)$ trace out the line segment S as t goes from 0 to 2.

$$x = t \qquad y = 2t$$

$$dx = dt \qquad dy = 2dt$$

$$I = \int_S (y^3+2x)dx + 3xy^2 dy$$

$$= \int_0^1 (8t^3+2t)dt + (12t^3).2dt$$

$$= \int_0^1 32t^3 + 2t \, dt$$

$$= 8t^4 + t^2 \Big|_0^1 = 9$$

<u>Problem</u> Evaluate

$$I = \int_S (6xy^2 - y^3)\, dx + (6x^2y - 3xy^2)\, dy$$

where S is <u>some</u> curve from $P = (1,2)$ to $Q = (3,4)$.

<u>Sol^n</u> Try to find $w = F(x,y)$ such that

$$dw = F_x \, dx + F_y \, dy =$$
$$(6xy^2 - y^3)\,dx + (6x^2y - 3xy^2)\,dy$$

well

$$F(x,y) = 3x^2y^2 - xy^3 + g(y)$$

$$F(x,y) = 3x^2y^2 - xy^3 + h(x)$$

we conclude that

$$F(x,y) = 3x^2y^2 - xy^3 + c$$

with c a constant.

so

$$I = \int_S d\left(3x^2y^2 - xy^3 + c\right)$$

$$= \int_{\partial S} 3x^2y^2 - xy^3 + c$$

$$= 3x^2y^2 - xy^3 + c \Big|_{(1,2)}^{(3,4)} = 236$$

To prove the above version of the fundamental Theorem of Calculus we need a bit about:

Partial Derivatives of Composite Functions

If

$$u = F(x_1, x_2, \cdots, x_n)$$

where

$$x_1 = g_1(r_1, r_2, \cdots, r_p)$$

$$x_2 = g_2(r_1, r_2, \cdots, r_p)$$

$$\vdots$$

$$x_n = g_n(r_1, r_2, \cdots, r_p)$$

then

$$\frac{\partial u}{\partial r_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial r_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial r_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial r_i}$$

Example Let

$n=2$

$p=1$

$$u = x^2 e^{yx}$$

$$x = t \cos t$$

$$y = t \sin t$$

find $\frac{\partial u}{\partial t}$ at $t = \pi/2$.

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}$$

$$= \left(x^2 y e^{yx} + 2x e^{yx}\right)\left(-t\sin t + \cos t\right)$$

$$+ \left(x^3 e^{yx}\right)\left(t\cos t + \sin t\right)$$

Evaluate this at $t = \pi/2$ to get $-\frac{\pi^3}{8}$.

Proof of fund. Theorem $(n=2)$

Suppose $\omega = F(x,y)$.

suppose $x = g(t)$, $y = h(t)$

is a parametrization of S

as t goes from t_0 to t_1.

$$\int_S d\omega = \int_S F_x\,dx + F_y\,dy$$

$$= \int_{t_0}^{t_1} F_x(g(t),h(t))\,\frac{dg}{dt}\,dt + F_y(g(t),h(t))\,\frac{dh}{dt}\,dt$$

$$= \int_{t_0}^{t_1} \frac{dF}{dt}\,dt = F(g(t_1),h(t_1)) - F(g(t_0),h(t_0))$$

$$= F(Q) - F(P).$$