Continuity, Differentiability and partial derivates

A function $f(x,y)$ is <u>continuous</u> if a small change in input only ever produces a small change in output.

More formally, $f(x,y)$ is <u>continuous</u> at a point (x_0, y_0) if, for any $\varepsilon > 0$, we can find a δ such that

i) $f(x,y)$ is defined and

ii) $|f(x,y) - f(x_0,y_0)| < \varepsilon$

when $|x - x_0| < \delta$ and $|y - y_0| < \delta$.

Example Consider

$$f(x,y) = \begin{cases} 3xy & , \quad (x,y) \neq (1,2) \\ 0 & , \quad (x,y) = (1,2) \end{cases}$$

well

$$\lim_{(x,y)\to(1,2)} f(x,y) = 6$$

and

$$f(1,2) = 0$$

Then $f(x,y)$ is not continuous at $(1,2)$.

Example Consider

$$f(x,y) = \begin{cases} \dfrac{x^2-y^2}{x^2+y^2} & , \quad (x,y) \neq (0,0) \\ 0 & , \quad (x,y) = (0,0) \end{cases}$$

Choose some constant m.

Suppose $x \to 0$, $y = mx \to 0$.

$$\lim_{\substack{x \to 0 \\ y = mx \to 0}} f(x,y) = \lim_{\substack{x \to 0 \\ y = mx \to 0}} \frac{x^2 - y^2}{x^2 + y^2}$$

$$= \lim_{x \to 0} \frac{x^2 - m^2 x^2}{x^2 + m^2 x^2}$$

$$= \frac{1 - m^2}{1 + m^2}$$

So $\lim_{\substack{x \to 0 \\ y \to 0}} f(x,y)$ does not exist

Hence $f(x,y)$ is **not** continuous at $(0,0)$.

Defn If a function $f(x,y)$ has continuous partial derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ in a region S then f is said to be **continuously differentiable**.

<u>Proposition</u> If f is continuously differentiable then it is i) continuous and ii) differentiable.

<u>Example</u> Consider

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Show that $f_x(0,0)$ and $f_y(0,0)$ are defined, but that $f(x,y)$ is not continuous at $(0,0)$.

<u>Sol^n</u>

$$f_x(0,0) = \lim_{h \to 0} \frac{f(0+h,0) - f(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{0}{h} = 0 \quad \text{exists}$$

Similarly

$$f_y(0,0) = \lim_{h \to 0} \frac{f(0,0+h) - f(0,0)}{h} = 0$$

exists.

Now, for any constant m,

$$\lim_{\substack{x \to 0 \\ y = mx \to 0}} f(x,y) = \lim_{x \to 0} \frac{mx^2}{x^2 + m^2x^2}$$

$$= \lim_{x \to 0} \frac{m}{1+m^2}$$

$$= \frac{m}{1+m^2}.$$

Since this depends on m we conclude that

$\lim\limits_{(x,y) \to (0,0)} f(x,y)$ does not exist.

Hence $f(x,y)$ is not continuous at $(0,0)$.