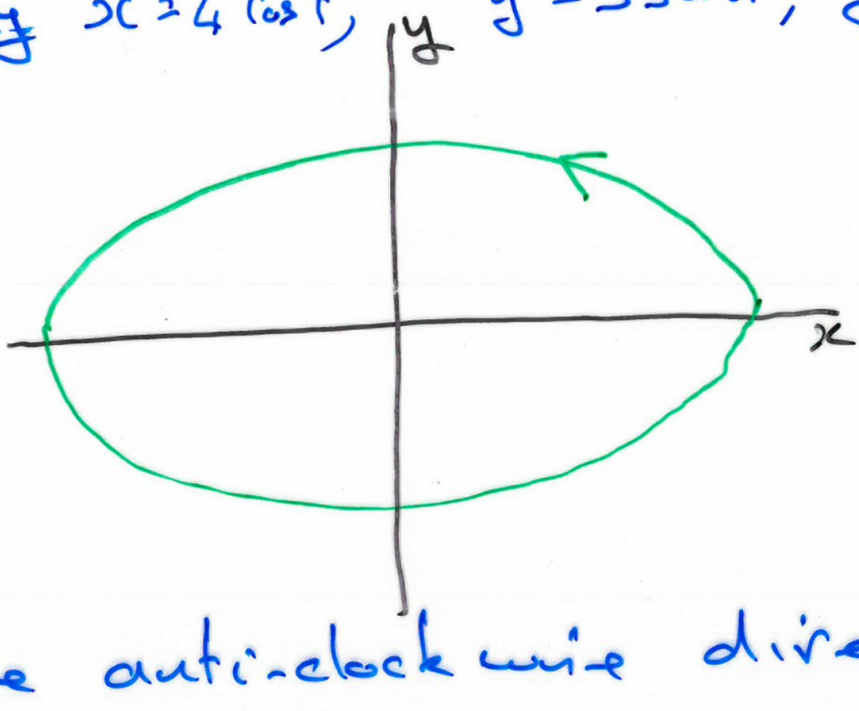
<u>Example</u> work is represented by the 1-form

$$\omega = (3x - 4y + 2z)\,dx + (4x + 2y - 3y^2)\,dy + (2xz - 4y^2 + z^3)\,dz.$$

Find the work done in moving a particle once around an ellipse C in the xy-plane with centre at the origin, and semi-major and semi-minor axes 5 and 4, given by $x = 4\cos t$, $y = 3\sin t$, $z = 0$.

<u>Sol^n</u>



Assume anti-clockwise direction.

Now C is given by

$$x = 4\cos t, \quad y = 3\sin t, \quad z = 0$$

work =

$$\int_C (3x-4y)\,dx + (4x+2y)\,dy$$

$$x = 4\cos t$$
$$dx = -4\sin t\,dt$$
$$y = 3\sin t$$
$$dy = 3\cos t\,dt$$

$$\int_{t=0}^{2\pi} -\big(3(4\cos t) - 4(3\sin t)\big)\,4\sin t\,dt + \big(4(4\cos t) + 2(3\sin t)\big)\,3\cos t\,dt$$

$$\ldots = \int_{t=0}^{2\pi} (48 - 30\sin t\cos t)\,dt$$

$$\ldots = 48t - 15\sin^2 t\,\Big|_0^{2\pi} = 96\pi$$

Partial Derivatives

Given a 0-form $\omega = f(x, y, z)$ we denote by

$$\frac{\partial f}{\partial x}$$

the 0-form got by regarding y, z as constants and then differentiating f with respect to x. We call $\frac{\partial f}{\partial x}$ the partial derivative of f with respect to x.

Example Consider the function

$$f(x, y, z) = \sqrt{1 - (x^2 + y^2 + z^2)}$$

defined on $S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < 1\}$

Calculate $\frac{\partial f}{\partial x}$.

Sol^n

$$f(x,y,z) = \left(1-(x^2+y^2+z^2)\right)^{\frac{1}{2}}$$

$$\frac{\partial f}{\partial x} = \frac{1}{2}\left(1-(x^2+y^2+z^2)\right)^{-\frac{1}{2}}(-2x)$$

$$= \frac{x}{\sqrt{x^2+y^2+y^2-1}} .$$

Similarly

$$\frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2+y^2+z^2-1}}$$

$$\frac{\partial f}{\partial z} = \frac{z}{\sqrt{x^2+y^2+z^2-1}}$$

Notation we often write f_x in place of $\frac{\partial f}{\partial x}$.

Differentiation of 0-forms

Given a 0-form $\omega = f(x, y, z)$ we define the 1-form

$$d\omega = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$

we call $d\omega$ the exterior derivative of ω. Sometimes we say that $d\omega$ is the total derivative of ω, or the differential.

Example Find the exterior derivative of the 0-form

$$\omega = \sqrt{1 - (x^2 + y^2 + z^2)}$$

on $S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\}$.

Sol 4

$$dw = \frac{x}{\sqrt{x^2+y^2+z^2-1}}\,dx + \frac{y}{\sqrt{x^2+y^2+z^2-1}}\,dy$$

$$+ \frac{z}{\sqrt{x^2+y^2+z^2-1}}\,dz$$