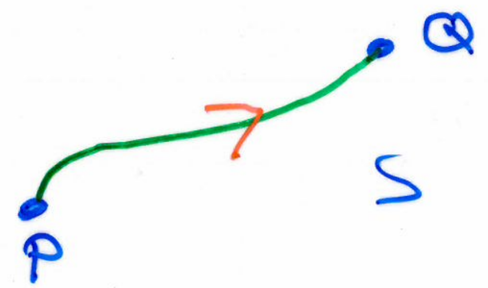
Integration of 1-forms

Let

$$\omega = A(x,y)\,dx + B(x,y)\,dy$$

be a 1-form.

Let $S \subseteq \mathbb{R}^2$ be a 1-dimensional oriented connected subspace.



<u>Informally</u>: if we think of ω as "work done" then

$$\int_S A(x,y)\,dx + B(x,y)\,dy$$

is the work done in moving a particle from P to Q along S.

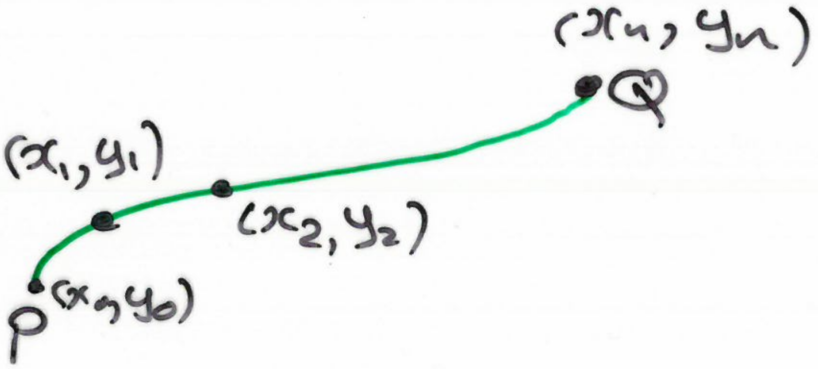
Formally

$$\int_S A(x,y)\,dx + B(x,y)\,dy =$$

$$\lim_{\|P\|\to 0} \sum_{i=1}^{n} A(x_i,y_i)(x_i-x_{i-1}) + B(x_i,y_i)(y_i-y_{i-1})$$

where

- $P = \{(x_0,y_0),(x_1,y_1),\dots,(x_n,y_n)\}$ is a sequence of points on S



with $(x_0,y_0) = P$ and $(x_n,y_n) = Q$

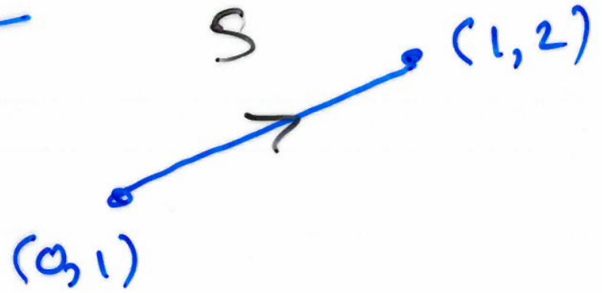
- $\|P\| = \max_{1\le i\le n} \|(x_i,y_i) - (x_{i-1},y_{i-1})\|$

with $\|(x,y)\| = \sqrt{x^2+y^2}$

Example Let S be the line segment from $P=(0,1)$ to $Q=(1,2)$.
Evaluate

$$\int_S (x^2-y)\,dx + (y^2+x)\,dy$$

Sol^n



The line $y=x+1$ passes through $(0,1)$ and $(1,2)$.

$$P=\{(x_0,y_0),(x_1,y_1),\dots (x_n,y_n)\}$$

has the form

$$P=\{(x_0,x_0+1),(x_1,x_1+1),\dots,(x_n,x_n+1)\}$$

$$L = \lim_{\|P\| \to 0} \sum_{i=1}^{m} (x_i^2 - (x_i + 1))(x_i - x_{i-1})$$

$$+ ((x_i + 1)^2 + x_i)(x_i - x_{i-1})$$

$$= \int_0^1 x^2 - (x+1) + ((x+1)^2 + x) \; dx$$

$$= \text{etc.}$$

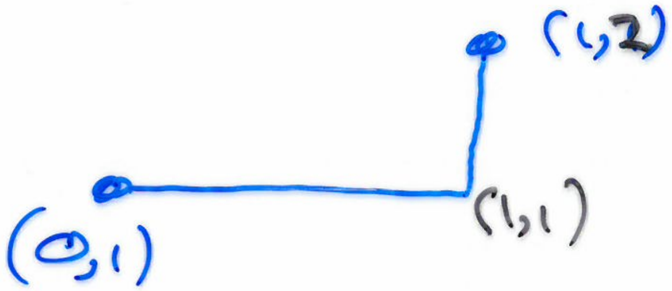
$$= \frac{5}{3} \, .$$

Problem Evaluate

$$\int_S (x^2 - y)\, dx + (y^2 + x)\, dy$$

where S is the line from $(0,1)$ to $(1,1)$ followed by the line from $(1,1)$ to $(1,2)$.

Soln



The line $y=1$ passes through $(0,1)$ & $(1,1)$

The line $x=1$ passes through $(1,1)$ & $(1,2)$

Integral =

$$\lim_{\|P\|\to 0} \sum_{i=1}^{n} (x_i^2-1)(x_i-x_{i-1}) + (1^2+x_i)\,0$$

$$+\lim_{\|P\|\to 0} \sum_{i=1}^{n} (1^2-y_i)\,0 + (y_i^2+1)(y_i-y_{i-1})$$

$$= \int_0^1 (x^2-1)\,dx + \int_1^2 (y^2+1)\,dy$$

$$= \frac{x^3}{3}-x\Big|_0^1 + \frac{y^3}{3}+y\Big|_1^2 = \frac{8}{3}.$$