Differential 1-forms on n-dimensional space

A differential 1-form on 2-dimensional space is a real valued function

$$\omega = A(x,y)\, h_1 + B(x,y)\, h_2$$

that inputs a vector (x,y) and a vector (h_1,h_2) and returns the number

$$A(x,y)\, h_1 + B(x,y)\, h_2 .$$

Here $A(x,y)$ and $B(x,y)$ are real valued diff. functions of x,y.

Example Evaluate the 1-form

$$\omega = (x^2+y^2)\, h_1 + 2xy\, h_2$$

at $(x,y) = (2,4)$ and $(h_1,h_2) = (0.25, 0.25)$.

Sol^n

Answer $= (2^2+4^2)0.25 + 2(2)(4)(0.25)$

$= 5 + 4 = 9$

Notation we usually write

$$A(x,y)\,dx + B(x,y)\,dy$$

instead of

$$A(x,y)\,h_1 + B(x,y)\,h_2$$

Example Evaluate the 1-form

$$\omega = (x^2+y^2)\,dx + 2xy\,dy$$

at $(x,y) = (2,4)$ and $(h_1,h_2) = (0.25, 0.25)$.

Sol^n

Answer $= 9$

Example A particle is moved in a constant force field. it takes 3 units of work to move the particle from (x,y) to $(x+1,y)$.
it takes 4 units of work to move the particle from (x,y) to $(x,y+1)$

We say that the work is represented by the differential 1-form

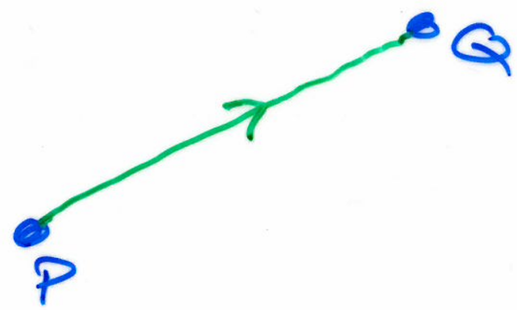
$$\text{work} = 3\,dx + 4\,dy$$

Example Consider a particle in a constant force field with work given by the 1-form

$$\omega = 2\,dx + 3\,dy + 5\,dz$$

Calculate the work done in moving a particle along the straight line segment from $P = (-1, 3, -5)$ to $Q = (3, -1, -7)$ in $\mathbb{R}^3$.

Sol^n



$Q - P = (3, -1, -7) - (-1, 3, -5) = (4, -4, -2)$

Required work =

$$2(4) + 3(-4) + 5(-2) = -14$$

Example If the force field is constant and if displacement of a particle

- from $(0,0,0)$ to $(4,0,0)$ needs 3 units of work
- from $(1,-1,0)$ to $(1,1,0)$ " 2 " "
- from $(0,0,0)$ to $(3,0,2)$ " 5 " "

then find the 1-form describing the function "work".

Soln

$$\omega = A\,dx + B\,dy + C\,dz$$

$3 = A(4)$ $\qquad A = \frac{3}{4}$

$2 = B(2)$ $\qquad B = 1$

$5 = A(3) + C(2)$

$$C = \frac{1}{2}(5 - 3A)$$

$$C = \frac{1}{2}\left(\frac{20}{4} - \frac{9}{4}\right) = \frac{11}{8}$$

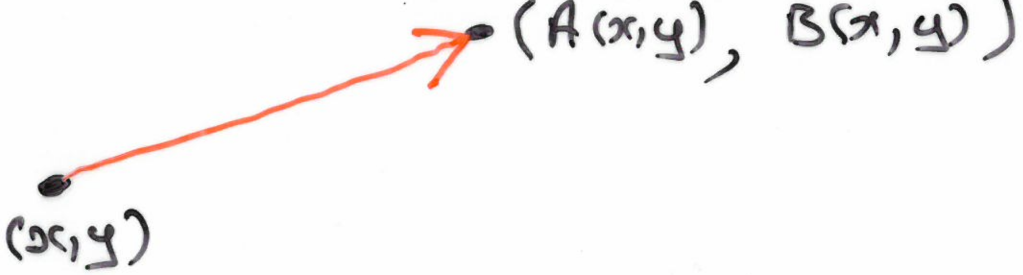
Required 1-form is

$$\omega = \frac{3}{4}dx + dy + \frac{11}{8}dz$$

we should think of a 1-form

$$\omega = A(x,y)\,dx + B(x,y)\,dy$$

as a collection of arrows in the plane. For each point (x,y) we have an arrow



<u>Example</u> The 1-form

$$\omega = 2\,dx + dy$$

can be pictured as

Example The 1-form

$$\omega = x\,dx + y\,dy$$

can be pictured as