Differential 1-forms in 1 variable

A differential 1-form (or 1-form) is a function of the form

$$\omega = f(x)\, h$$

which inputs two numbers $x, h \in \mathbb{R}$ and returns the number $f(x)h$.

Here $f(x)$ has to be a differentiable function of x.

Example Evaluate the 1-form

$$\omega = (x^2+6)\, h$$

at $x=2$, $h=0.5$.

Answer: $(2^2+6)\cdot 0.5 = 5$

Notation we usually denote the 1-form $f(x)h$ by

$$f(x)\, dx$$

Example Evaluate the 1-form

$$\omega = \sin(x)\,dx$$

at $x = \frac{\pi}{2}$, $h = 0.25$.

Answer: $\sin(\frac{\pi}{2})\, 0.25 = 0.25$

Defn Given a 1-form

$$\omega = f(x)\,dx$$

and an oriented interval

$$S = [a, b]$$

we define the integral

$$\int_S \omega = \int_a^b f(x)\,dx$$

where the right-hand side was defined in 1st year.

<u>Informally</u>: $\int_a^b f(x)\,dx$ is the area between the curve of $y=f(x)$ and the x-axis from a to b where, if $a<b$, areas above x-axis are positive and below x-axis are negative.

<u>Formally</u>

$$\int_a^b f(x)\,dx = \lim_{\|P\|\to 0} \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1})$$

where:

$P = \{a = x_0, x_1, x_2, \dots, x_n = b\}$ is a sequence of numbers in $[a,b]$

$$\|P\| = \max_{1\le i\le n} |x_i - x_{i-1}|$$

$$\xi_i \in [x_{i-1}, x_i]$$

Fundamental Theorem of Calculus

For a 0-form $\omega = F(x)$ defined on $[a, b]$ with $\frac{d}{dx}F(x) = f(x)$ we have

$$\int_{\partial[a,b]} F(x) = \int_{[a,b]} f(x)\, dx$$

Example Integrate the 1-form

$$\omega = (3x^2 + 2x)\, dx$$

on the interval $[3, 0]$.

Sol^n

Consider the 0-form

$$F(x) = x^3 + x^2$$

so $\frac{d}{dx}F(x) = 3x^2 + 2x$

So $\int_{[3,0]} f(x)\,dx \underset{\text{FTC}}{=} \int_{\partial[3,0]} x^3 + x^2$

$$= F(0) - F(3) = 0 - [3^3 + 3^2]$$

$$= -36$$

Explanation/proof of FTC

Suppose Galway to Dublin train has a functioning speedometer but broken mileometer.

To estimate the distance travelled from time $t = a$ to $t = b$ the driver could calculate

$$\sum_{i=1}^{n} f(t_i)(t_i - t_{i-1})$$

where $f(t)$ = speed/velocity of train at time t.

Let $F(t)$ = distance travelled from initial time to time t.

$$\frac{d}{dt} F(t) = f(t)$$

Roughly

$$F(b) - F(a) \approx \sum_{i=1}^{n} f(t_i)(t_i - t_{i-1})$$

Taking limits yields

$$F(b) - F(a) = \int_a^b f(t)\, dt.$$