Curl - alternative definition

Given a vector field

$$F = F_1 \underline{i} + F_2 \underline{j} + F_3 \underline{k}$$

on $\mathbb{R}^3$ we define the associated

work 1-form

$$\omega = F_1\, dx + F_2\, dy + F_3\, dz$$

Suppose the exterior derivative of ω is

$$d\omega = A\, dx \wedge dy + B\, dy \wedge dz + C\, dz \wedge dx$$

Then we define

$$\mathrm{Curl}(F) = B \underline{i} + C \underline{j} + A \underline{k}$$

Example $F = xz\, \underline{i} + y^2\, \underline{j} + 2x^2 y\, \underline{k}$

$$\omega = xz\, dx + y^2\, dy + 2x^2 y\, dz$$

$$d\omega = (z\, dx + x\, dz) \wedge dx + (2y\, dy) \wedge dy$$

$$+ (4xy\, dx + 2x^2 dy) \wedge dz$$

$$d\omega = x\, dz \wedge dx + 4xy\, dx \wedge dz + 2x^2\, dy \wedge dz$$

$$d\omega = 0\, dx \wedge dy + 2x^2\, dy \wedge dz + (x - 4xy)\, dz \wedge dx$$

$$\operatorname{Curl}(\underline{F}) = 2x^2\, \underline{\underline{i}} + (x - 4xy)\, \underline{\underline{j}}$$

Interpretation of curl

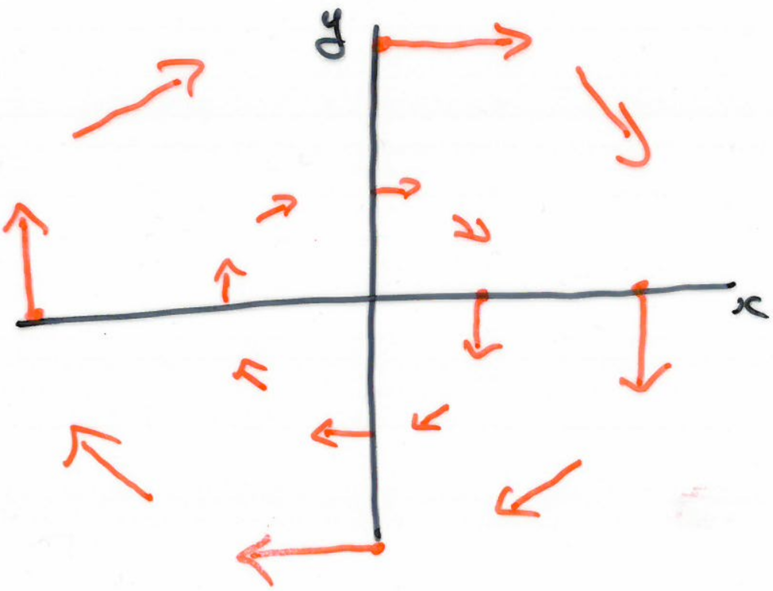
Imagine a vector field $\underline{F}$ on $\mathbb{R}^3$ describes the directions and speeds of particles in a fluid.

Place a small rough spherical ball in the fluid, fixed at point (x, y, z). The fluid will make the ball rotate. The axis of rotation is the direction of the vector $\operatorname{Curl}(\underline{F})$. The angular speed of rotation is given by the size of $\operatorname{Curl}(\underline{F})$.

<u>Example</u> Consider

$$F(x,y,z) = y\,\underline{\underline{i}} - x\,\underline{\underline{j}}$$

In the xy-plane we have



$$\omega = y\,dx - x\,dy$$

$$d\omega = dy \wedge dx - dx \wedge dy = -2\,dx \wedge dy$$

$$\mathrm{Curl}(F) = -2\,\underline{\underline{k}}$$

So a ball rotates at speed 2, clockwise, about an axis parallel to the z-axis, no matter where we place the ball.

Divergence

Given a vector field

$$F = F_1 \underline{i} + F_2 \underline{j} + F_3 \underline{k}$$

on $\mathbb{R}^3$ we define the associated

flux 2-form

$$\omega = F_3 \, dx \wedge dy + F_1 \, dy \wedge dz + F_2 \, dz \wedge dy$$

The exterior derivative of ω

is a 3-form

$$d\omega = A \, dx \wedge dy \wedge dz$$

Defn We define the divergence

of F to be the function

$$\operatorname{div}(F) = A$$

Example $\underline{F} = xz\,\underline{i} - y^2\,\underline{j} + 2x^2y\,\underline{k}$

$$\omega = 2x^2y \; dx \wedge dy + xz \; dy \wedge dz - y^2 dz \wedge dx$$

$$d\omega = (z\,dx) \wedge dy \wedge dz + (-2y\,dy) \wedge dz \wedge dx$$

$$= (z - 2y) \; dx \wedge dy \wedge dz$$

$$\operatorname{div}(F) = z - 2y$$