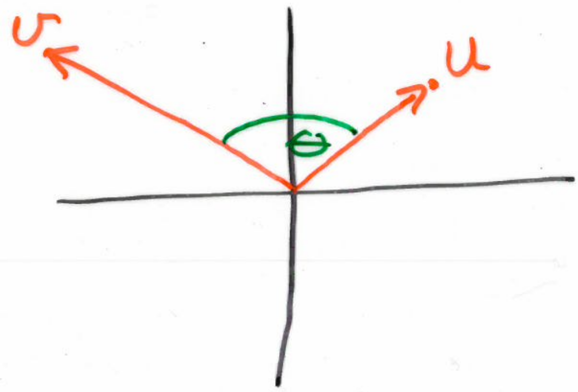
Dot Products of Vectors

Given two vectors

$$u = (u_1, u_2)$$

$$v = (v_1, v_2)$$

in the plane



We define their dot product to be the number

$$u \cdot v = u_1 v_1 + u_2 v_2$$

Example If $u = (2,3)$, $v = (4,5)$ then $u \cdot v = 2 \cdot 4 + 3 \cdot 5 = 23$

We define the length of u, v to be:

$$|u| = \sqrt{u_1^2 + u_2^2}$$

$$|v| = \sqrt{v_1^2 + v_2^2}$$

It is easy to prove the following.

<u>Theorem</u> $u.v = |u|\,|v| \cos(\theta)$

In particular, u and v are perpendicular to each other if and only if $u.v = 0$

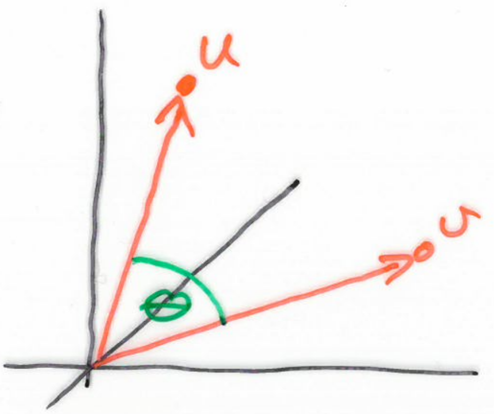
For two vectors

$u = (u_1, u_2, u_3)$

$v = (v_1, v_2, v_3)$

in $\mathbb{R}^3$ we define the dot product

$$u.v = u_1 v_1 + u_2 v_2 + u_3 v_3$$



Again:

<u>Theorem</u>

$u.v = |u|.|v|. \cos(\theta)$

In particular, u, v are at right-angles if and only if $u.v = 0$

Div, Grad, Curl and all that

Gradient

Let $\phi(x,y,z)$ be a real valued differentiable function. The **gradient** of ϕ is defined as:

$$\text{grad}\,\phi = \nabla\phi = \frac{\partial\phi}{\partial x}\underline{\underline{i}} + \frac{\partial\phi}{\partial y}\underline{\underline{j}} + \frac{\partial\phi}{\partial z}\underline{\underline{k}}$$

where

$$\underline{\underline{i}} = (1,0,0)$$

$$\underline{\underline{j}} = (0,1,0)$$

$$\underline{\underline{k}} = (0,0,1).$$

often we'll forget to do the underlining

We pronounce $\nabla\phi$ as "del phi"

We could also think of the gradient of a 0-form $\omega = \phi$ as

$$\nabla\phi = d\phi$$

where we think of

$$\underline{\underline{i}} = dx$$
$$\underline{\underline{j}} = dy$$
$$\underline{\underline{k}} = dz$$

<u>Interpretation of the gradient (or exterior derivative of 0-forms)</u>

Consider a surface S defined by an equation

$$\phi(x, y, z) = k \qquad (k \text{ a constant})$$

<u>Example</u> Let $\phi(x, y, z) = x^2 + y^2 + z^2$
Let $k = 9$

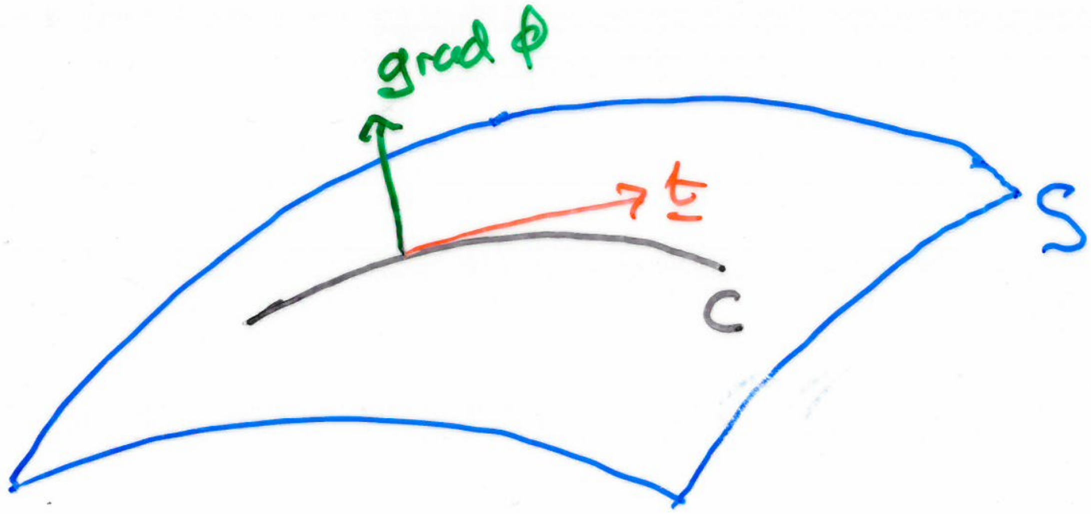
The equation

$$x^2 + y^2 + z^2 = 9$$

describes the sphere S of radius 3, centred at the origin. $\square$

Let C be a curve on our surface S parametrized as

$$C : \mathbb{R} \to S, \quad t \mapsto (x(t), y(t), z(t)) .$$



Note that

$$\phi(x(t), y(t), z(t)) = k$$

k the constant, whatever curve C we choose.

The chain rule gives

$$0 = \frac{\partial \phi}{\partial t} = \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial \phi}{\partial z}\frac{\partial z}{\partial t}$$

$$= \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right) \cdot \left(\frac{\partial x}{\partial t}, \frac{\partial y}{\partial t}, \frac{\partial z}{\partial t}\right)$$

Now

$$\left(\frac{\partial x}{\partial t}, \frac{\partial y}{\partial t}, \frac{\partial z}{\partial t}\right) = \frac{\partial x}{\partial t}\underline{\underline{i}} + \frac{\partial y}{\partial t}\underline{\underline{j}} + \frac{\partial z}{\partial t}\underline{\underline{k}}$$

is a tangent $\underline{t}$ to the curve C and to the surface S.

Thus

$$\left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right) = \frac{\partial\phi}{\partial x}\underline{\underline{i}} + \frac{\partial\phi}{\partial y}\underline{\underline{j}} + \frac{\partial\phi}{\partial z}\underline{\underline{k}}$$

is a vector perpendicular to (every) tangent $\underline{t}$ to the surface S.

In other words, grad ϕ is a vector, depending on x, y, z, which is perpendicular to the surface S at the point $(x, y, z) \in S$.