Example Calculate $d\omega$ for

$$\omega = x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy.$$

Sol^n

$$d\omega = dx \wedge dy \wedge dz + dy \wedge dz \wedge dx + dz \wedge dx \wedge dy$$

$$= dx \wedge dy \wedge dz - dy \wedge dx \wedge dz - dx \wedge dz \wedge dy$$

$$= dx \wedge dy \wedge dz + dx \wedge dy \wedge dz + dx \wedge dy \wedge dz$$

$$= 3\, dx \wedge dy \wedge dz$$

Proposition For any k-form ω we have

$$d(d\omega) = 0$$

<u>Example</u> Prove that

$$\omega = (3x^2 - 6yz)\,dx + (2y + 3xz)\,dy + (1 - 4xyz^2)\,dz$$

does <u>not</u> arise as $\omega = d\nu$ for any 0-form ν.

<u>Sol^n</u> It suffices to show that $d\omega \neq 0$.

$$d\omega = (6x\,dx - 6z\,dy - 6y\,dz)\wedge dx + (2\,dy + 3z\,dx + 3x\,dz)\wedge dy + (-4yz^2\,dx - 4xz^2\,dy - 8xyz\,dz)\wedge dz$$

$$= -6z\,dy\wedge dx - 6y\,dz\wedge dx + 3z\,dx\wedge dy + 3x\,dz\wedge dy - 4yz^2\,dx\wedge dz - 4xz^2\,dy\wedge dz$$

$= 6z\, dx \wedge dy + 3z\, dx \wedge dy + \cdots\cdots$

$= 9z\, dx \wedge dy + \cdots\cdots$

$\neq 0.$ Done

<u>Exercise</u> For 0-forms ν, ω we have

$$d(\nu\omega) = (d\nu)\omega + \nu(d\omega).$$

<u>Proof</u>

$$\frac{\partial}{\partial x}(\nu\omega)\, dx + \frac{\partial}{\partial y}(\nu\omega)\, dy + \cdots$$

$$= \left(\frac{\partial \nu}{\partial x}\omega + \nu\frac{\partial \omega}{\partial x}\right) dx + \left(\frac{\partial \nu}{\partial y}\omega + \nu\frac{\partial \omega}{\partial y}\right) dy + \cdots$$

$$= \left(\frac{\partial \nu}{\partial x} dx + \frac{\partial \nu}{\partial y} dy\right)\omega + \nu\left(\frac{\partial \omega}{\partial x} dx + \frac{\partial \omega}{\partial y} dx\right) + \cdots$$

$$= (d\nu)\,\omega + \nu(d\omega).$$

<u>Exercise</u> For 1-forms ν, ω we have

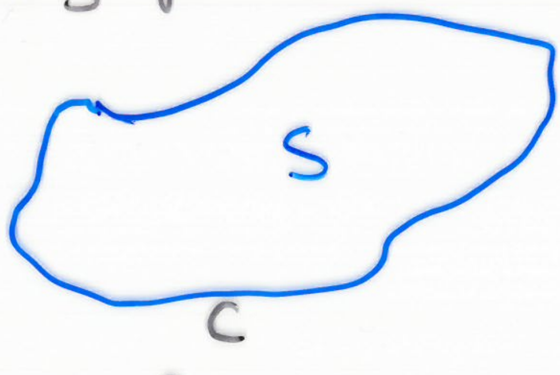
$$d(\nu \wedge \omega) = (d\nu)\wedge\omega - \nu(d\omega)$$

for you to try!

<u>Theorem</u> For a k-form ν and n-form ω we have

$$d(\nu \wedge \omega) = (d\nu)\wedge\omega + (-1)^{kn}\nu(d\omega)$$

<u>Example</u> Show that the area of the region S bounded by a simple closed curve $C = \partial S$ in the xy-plane



is $\frac{1}{2}\int_C x\,dy - y\,dx$.

Sol^n

$$\frac{1}{2}\int_{\partial S} x\,dy - y\,dx$$

$$= \frac{1}{2}\int_S d(x\,dy - y\,dx)$$

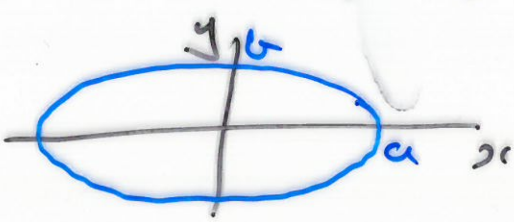
Stokes' (or Green's) formula

$$= \frac{1}{2}\int_S dx \wedge dy - dy \wedge dx$$

$$= \int_S dx \wedge dy$$

$=$ area of S (up to sign)

__Example__ Find the area of the region S bounded by the ellipse $x = a\cos\theta$, $y = b\cos\theta$.



Sol^n From above,

required area =

$$\frac{1}{2}\int_{\partial S} x\,dy - y\,dx$$

$$= \frac{1}{2}\int_0^{2\pi} (a\cos\theta)(b\cos\theta)\,d\theta - (b\sin\theta)(-a\sin\theta)\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} ab(\cos^2\theta + \sin^2\theta)\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} ab\,d\theta$$

$$= \frac{1}{2}\,ab\theta\Big|_0^{2\pi}$$

$$= ab\pi.$$