Differentiation of 1-forms (continued)

For a 1-form ω we have a 2-form $d\omega$. The definition of $d\omega$ is based on our understanding of integration of 2-forms, and is crafted to ensure that Stokes' formula

$$\int_{\partial S} \omega = \int_S \omega$$

holds under reasonable hypotheses.

The following computational rules hold:

For 1-forms ω and ω' and for functions $A, B, C, \dots$ in variables $x, y, z, \dots$

1. $d(\omega + \omega') = d\omega + d\omega'$ (linearity)

2. $dA = \frac{\partial A}{\partial x} dx + \frac{\partial A}{\partial y} dy + \frac{\partial A}{\partial z} dz + \cdots$

3. $d(A\,dx + B\,dy + \cdots) = (dA) \wedge dx + (dB) \wedge dy + \cdots$

4. $dx \wedge dx = 0, \quad dy \wedge dy = 0, \quad \dots$

5. $dx \wedge dy = -dy \wedge dx, \quad dx \wedge dz = -dz \wedge dx, \dots$

6. $(\omega + \omega') \wedge dx = \omega \wedge dx + \omega' \wedge dx, \quad \dots$

7. See later

Example Calculate $d\omega$ for

$$\omega = xy\,dz + yz\,dx + zx\,dy$$

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$$d\omega = d(xy\,dz) + d(yz\,dx) + d(zx\,dy)$$

$$= d(xy)\wedge dz + d(yz)\wedge dx + d(zx)\wedge dy$$

$$= (y\,dx + x\,dy)\wedge dz$$
$$+ (z\,dy + y\,dz)\wedge dx$$
$$+ (z\,dx + x\,dz)\wedge dy$$

$$= y\,dx\wedge dz + x\,dy\wedge dz$$
$$+ z\,dy\wedge dx + y\,dz\wedge dx$$
$$+ z\,dx\wedge dy + x\,dz\wedge dy$$

$$= y\,dx\wedge dz - y\,dx\wedge dz$$
$$+ x\,dy\wedge dz - x\,dy\wedge dz$$
$$+ z\,dx\wedge dy - z\,dx\wedge dy = 0$$

Proposition Let $\omega = F(x,y,z)$ be a 0-form. Suppose $F_{xy} = F_{yx}$, $F_{xz} = F_{zx}$, $F_{yz} = F_{zy}$. Then

$$d(d\omega) = 0$$

Proof

$$d(d\omega) = d(F_x\,dx + F_y\,dy + F_z\,dz)$$

$$= (F_{xx}\,dx + F_{xy}\,dy + F_{xz}\,dz) \wedge dx$$

$$+ (F_{yx}\,dx + F_{yy}\,dy + F_{yz}\,dz) \wedge dy$$

$$+ (F_{zx}\,dx + F_{zy}\,dy + F_{zz}\,dz) \wedge dz$$

$$= \ldots$$

$$= 0$$

Differentiation of k-forms

A 2-form is an expression such as

$$\omega = A\,dx \wedge dy + B\,dy \wedge dz + \cdots$$

where $A, B, \ldots$ are functions of $x, y, z, \ldots$

A 3-form is an expression such as

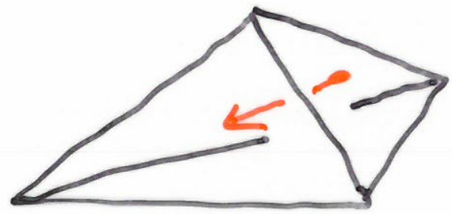
$$\omega = A\,dx \wedge dy \wedge dz + B\,dy \wedge dz \wedge dw + \cdots$$

To understand integrals of 2-forms we needed to understand how to integrate constant 2-forms over oriented planar triangles. (area)

For integrals of 3-forms we need to understand how to integrate constant 3-forms over

tetrahedra:

An orientation of such a tetrahedron can be specified by an arrow on its surface pointing either inwards or outwards:



Given a 2-form ω we define a 3-form $d\omega$ such that

$$\int_{\partial S} \omega = \int_S d\omega$$

where S is an oriented 3-dimensional region.

The derivate satisfies rules 1-6 above and

7. $(dx \wedge dy) \wedge dz = dx \wedge (dy \wedge dz)$

we usually just write

$$dx \wedge dy \wedge dz .$$

<u>Exercise</u> Calculate $d\omega$ for

$$\omega = x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy .$$