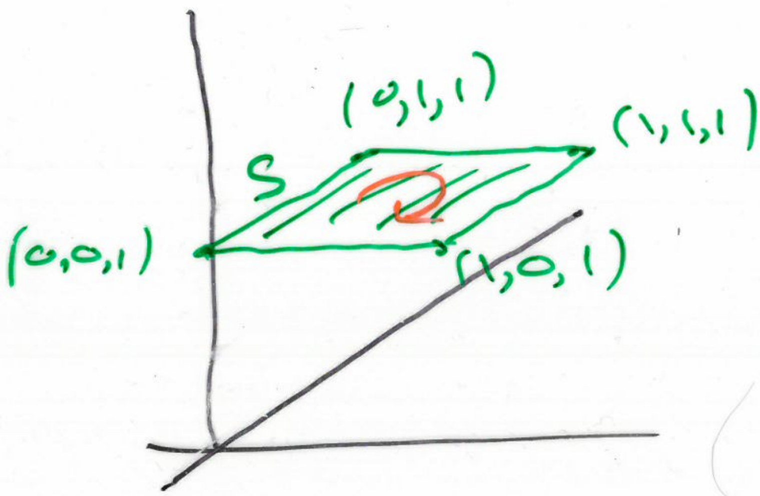
Example Evaluate

$$I = \int_S (x+y+z)\, dx \wedge dy$$

where S is the oriented planar rectangle with vertices $(0,0,1)$, $(0,1,1)$, $(1,1,1)$, $(1,0,1)$ in that order.

Sol^n

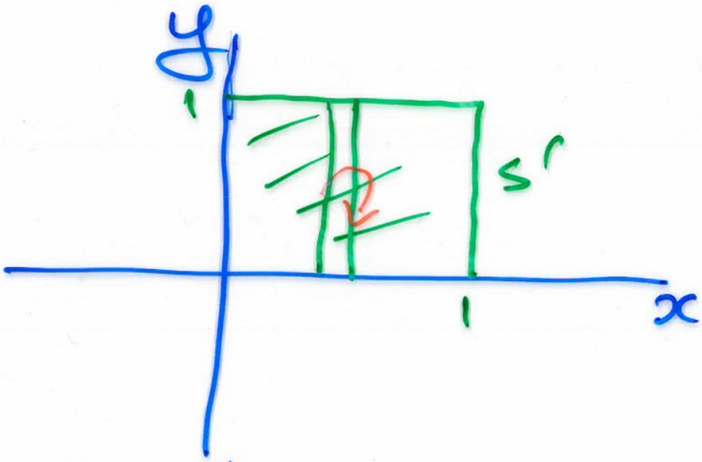


On S we have $z = 1$.

So

$$I = \int_{S'} (x+y+1)\, dx \wedge dy$$

where S' is the rectangle in xy-plane with vertices $(0,0)$, $(0,1)$, $(1,1)$, $(1,0)$



$$\pm I = \int_{x=0}^{x=1} \left(\int_{y=0}^{1} (x+y+1)\, dy \right) dx$$

$$= \int_{x=0}^{1} xy + \frac{y^2}{2} + y \Big|_0^1 \, dx$$

$$= \int_{x=0}^{1} x + \frac{3}{2} \, dx$$

$$= \frac{x^2}{2} + \frac{3}{2}x \Big|_0^1 = 2.$$

$I = -2$ due to the orientation

Differentiation of 1-forms

For a 1-form ω, and 2-dimensional oriented region S, we'd like to define a 2-form

$$d\omega$$

such that

$$\boxed{\int_{\partial S} \omega = \int_S d\omega \qquad (*)}$$

The definition of $d\omega$ is determined by the wish to have equation $(*)$ hold.

Suppose

$$\omega = A\,dx + B\,dy$$

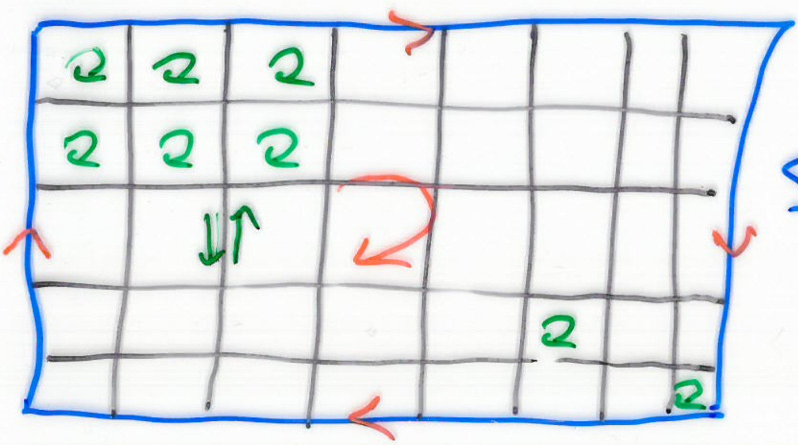
where A, B are functions of x, y.
We want to define

$$d\omega = C\,dx \wedge dy$$

such that

$$\boxed{\int_{\partial S} A\,dx + B\,dy = \int_S C\,dx \wedge dy}$$

For simplicity, suppose S is an oriented region in the xy-plane, with boundary ∂S oriented accordingly.



$$S = S_1 \cup S_2 \cup \dots \cup S_n$$

Note that

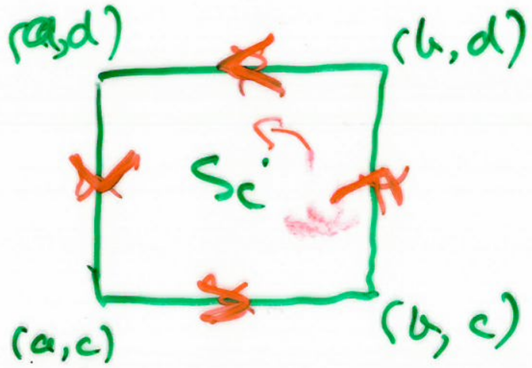
$$\int_{\partial S} A\,dx + B\,dy = \sum_{i=1}^{n} \int_{\partial S_i} A\,dx + B\,dy$$

So, for each small square S_i we just need

$$\boxed{\int_{\partial S_i} A\,dx + B\,dy = \int_{S_i} c\,dx \wedge dy} \quad (**)$$

Suppose S_i is the square

$$a \le x \le b$$
$$c \le y \le d$$



Assume such a $c\,dx \wedge dy$ exists.

We have

$$\int_{\partial S_i} A\,dx + B\,dy$$

$$= \int_a^b A(x,c)\,dx + \int_c^d B(b,y)\,dy$$

$$+ \int_b^a A(x,d)\,dx + \int_d^c B(a,y)\,dy$$

$$= \int_c^d \big(B(b,y) - B(a,y)\big)\,dy$$

$$- \int_a^b \big(A(x,d) - A(x,c)\big)\,dx$$

$$\overset{FTC}{=} \int_c^d \left[\int_a^b \frac{\partial B}{\partial x}\, dx\right] dy - \int_a^b \left[\int_c^d \frac{\partial A}{\partial y}\, dy\right] dx$$

$$= \int_{S_i} \frac{\partial B}{\partial x}\, dx \wedge dy - \int_{S_i} \frac{\partial A}{\partial y}\, dx \wedge dy$$

$$= \int_{S_i} \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right) dx \wedge dy$$

Thus

$$\boxed{d\omega = c\, dx \wedge dy = \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right) dx \wedge dy}$$

From a careful analysis of integrals one can show that $d\omega$ must satisfy the following computational rules.

For 1-forms ω, ω'

$$\boxed{d(\omega + \omega') = d\omega + d\omega'}$$

For functions $A, B, \dots$ in variables $x, y, z \dots$

$$\boxed{dA = \frac{\partial A}{\partial x} dx + \frac{\partial A}{\partial y} dy + \frac{\partial A}{\partial z} dz}$$

$$\boxed{d(A\,dx + B\,dy + \cdots) = dA \wedge dx + dB \wedge dy + \cdots}$$

$$\boxed{dx \wedge dx = 0}$$

$$\boxed{dx \wedge dy = -dy \wedge dx}$$

Example for

$$\omega = A\,dx + B\,dy$$

$$d\omega = d(A\,dx + B\,dy)$$

$$= (dA)\wedge dx + (dB)\wedge dy$$

$$= \left(\frac{\partial A}{\partial x}dx + \frac{\partial A}{\partial y}dy\right)\wedge dx$$

$$+ \left(\frac{\partial B}{\partial x}dx + \frac{\partial B}{\partial y}dy\right)\wedge dy$$

$$= \cancel{\frac{\partial A}{\partial x}dx\wedge dx}^{=0} + \frac{\partial A}{\partial y}dy\wedge dx$$

$$+ \frac{\partial B}{\partial x}dx\wedge dy + \cancel{\frac{\partial B}{\partial y}dy\wedge dy}^{=0}$$

$$= \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right)dx\wedge dy$$