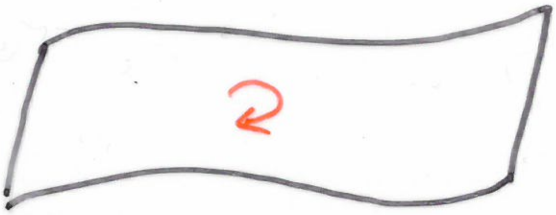
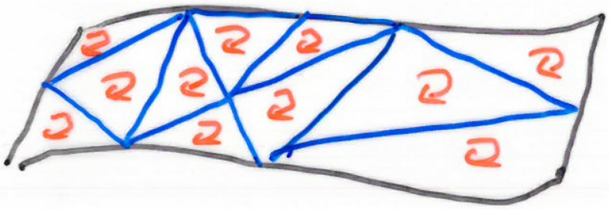
Integration of 2-forms

Let S denote a 2-dimensional region in $\mathbb{R}^3$ with some choice of orientation.



We can approximate S by a union of oriented planar triangles



$P = T_1 \cup T_2 \cup \cdots \cup T_k$

union of k oriented triangles

Suppose we have a sequence of approximations $P_1, P_2, P_3, \ldots$ where:

1) the approximation P_i gets better as $i \to \infty$
2) the area of the largest triangle in P_i tends to 0 as $i \to \infty$.

We define:

$$\int_S A(x,y,z)\,dx\wedge dy + B(x,y,z)\,dy\wedge dz + C(x,y,z)\,dz\wedge dx$$

$$= \lim_{i\to\infty} \sum_{T_j \text{ in } P_i} \int_{T_j} A(x_j,y_j,z_j)\,dx\wedge dy + B(x_j,y_j,z_j)\,dy\wedge dz + C(x_j,y_j,z_j)\,dz\wedge dx$$

where the point (x_j, y_j, z_j) lies in T_j

<u>Example</u> Evaluate

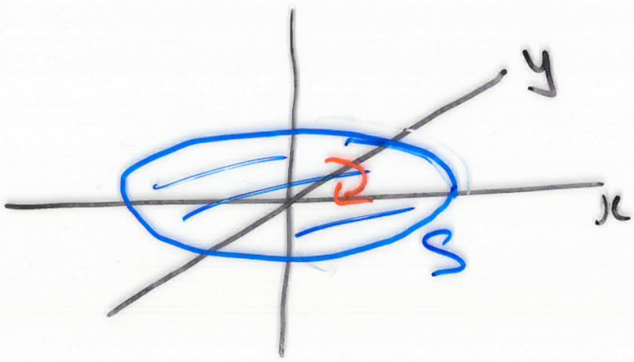
$$I = \int_S 3\,dx\wedge dy + 4\,dy\wedge dz$$

where S is the disk

$$S = \{(x,y,z)\in\mathbb{R}^3 : z=0,\ x^2+y^2\le 1\}$$

with clockwise orientation.

<u>Sol^n</u>



From the definition we see that

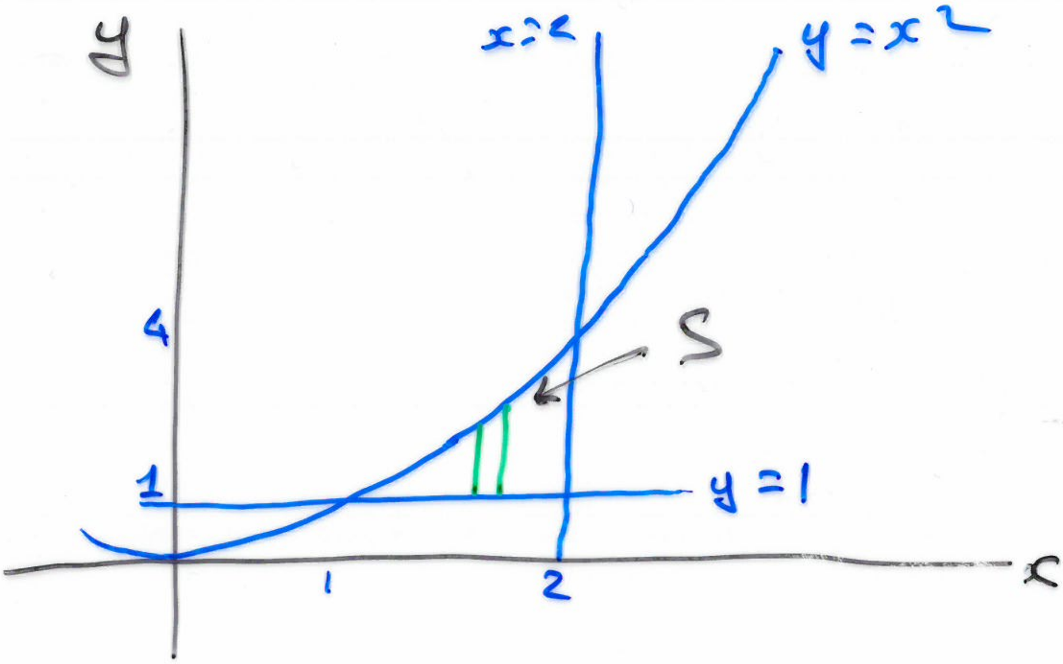
$$I = 3 \times (-1) \text{ area of disk} = -3\pi$$

<u>Example</u> Let S be the region in the xy-plane bounded by $y = x^2$, $x = 2$, $y = 1$. Let S have an anti-clockwise orientation, and evaluate

$$I = \int_S (x^2+y^2+z^2)\, dx \wedge dy$$

<u>Sol^n</u> On S we have $z = 0$ and thus

$$I = \int_S (x^2+y^2)\, dx \wedge dy$$



Subdivide S into thin strips parallel to y-axis

We can write

$$I = \int_{x=1}^{x=2} \left(\int_{y=1}^{y=x^2} (x^2+y^2)\, dy \right) dx$$

$$I = \int_{x=1}^{x=2} \left(x^2 y + \frac{y^3}{3} \Bigg|_{y=1}^{y=x^2} \right) dx$$

$$I = \int_{x=1}^{x=2} \left(x^4 + \frac{x^6}{3} - x^2 - \frac{1}{3} \right) dx$$

$$I = \left. \frac{x^5}{5} + \frac{x^7}{21} - \frac{x^3}{3} - \frac{x}{3} \right|_1^2 = \frac{1006}{105}$$