Protest! Strike? Wednesday

Move the first test to Monday?.

Continuity

A function $f(x,y)$ is continuous at (x_0, y_0) if

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y)$$

exists, and

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0).$$

Example Consider

$$f(x,y) = \begin{cases} 3xy & (x,y) \neq (1,2) \\ 0 & (x,y) = (1,2) \end{cases}$$

At the point $(1,2)$

$$\lim_{(x,y)\to(1,2)} f(x,y) = \lim_{(x,y)\to(1,2)} 3xy = 6$$

$$f(1,2) = 0 .$$

Hence $f(x,y)$ is **not** continuous at $(1,2)$.

Example Consider

$$f(x,y) = \begin{cases} \dfrac{x^2-y^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Choose some constant m. 3

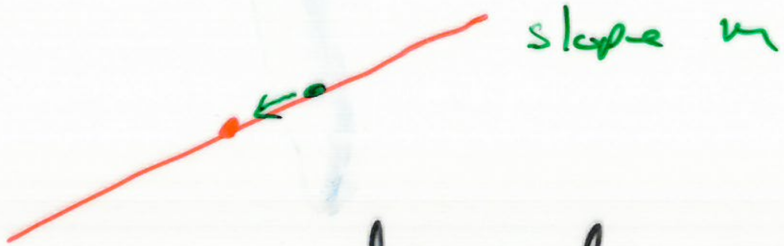
Suppose $x \to 0$. Then $y = mx \to 0$.

$$\lim_{(x,y)\to(0,0)} f(x,y) = ?$$

$$\lim_{\substack{x\to 0\\ y=mx\to 0}} f(x,y) =$$

$$\lim_{x\to 0} \frac{x^2 - (mx)^2}{x^2 + (mx)^2}$$

$$= \frac{1-m^2}{1+m^2}.$$



This answer depends on m. Then $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist, it follows that $f(x,y)$ is not continuous at $(0,0)$.

Definition If a function $f(x,y)$ has continuous partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ in a region S, then f is said to be *continuously differentiable* in the region.

Proposition If f is continuously differentiable in a region then f is continuous in the region, and f is differentiable in the region.

Example Consider

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0). \end{cases}$$

Show that $f_x(0,0)$ exists, and that $f_y(0,0)$ but that $f(x,y)$ is not continuous at $(0,0)$.

Sol

$f_x(0,0) =$

$$\lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{0}{h} = 0$$

$f_y(0,0) =$

$$\lim_{h \to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h \to 0} \frac{0}{h} = 0.$$

Hence $f_x(0,0)$ and $f_y(0,0)$ exist.

Consider $y = mx$

$$\lim_{\substack{x \to 0 \\ y = mx \to 0}} \frac{xy}{x^2+y^2} = \lim_{x \to 0} \frac{mx^2}{(1+m^2)x^2}$$

$$= \frac{m}{1+m^2}.$$

Hence

$\lim_{(x,y)\to(0,0)} f(x,y)$ does <u>not</u> exist.

Hence $f(x,y)$ is <u>not</u> continuous

at $(0,0)$.

Partial derivatives of composite functions

(Chain Rule)

If

$$u = F(x_1, x_2, \cdots, x_n)$$

where

$$x_1 = g_1(r_1, r_2, \cdots, r_p)$$

$$x_2 = g_2(r_1, r_2, \cdots, r_p)$$

$$\vdots$$

$$x_n = g_n(r_1, r_2, \cdots, r_p),$$

then

$$\frac{\partial u}{\partial r_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial r_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial r_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial r_i}$$

Proof is easy & boring.

Example Consider

$$u = x^2 e^{yx}$$

where

$$x = t\cos(t)$$
$$y = t\sin(t).$$

find

$$\frac{du}{dt} \quad \text{at} \quad t = \frac{\pi}{2}.$$

Sol^n

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}$$

$$= \left(2x e^{yx} + x^2 e^{yx} y\right)\left(\cos(t) - t\sin(t)\right)$$

$$+$$

$$x^3 e^{yx}\left(\sin(t) + t\cos(t)\right)$$

Evaluate at $t = \frac{\pi}{2}$:

$$\left.\frac{\partial u}{\partial t}\right|_{t=\frac{\pi}{2}} = \text{etc.}$$