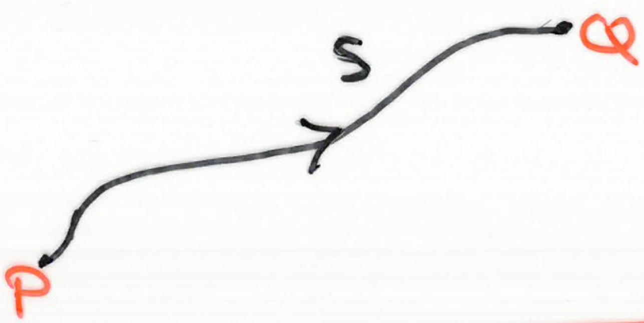
MA2286 Test : 4 October 2017

All material up to and including the definition of total derivatives.

Fundamental Theorem of Calculus

Let ω be a 0-form on n-dimensional space.

Let S be a curve in $\mathbb{R}^n$ from P to Q.



Theorem

$$\int_S d\omega = \int_{\partial S} \omega$$

Example Evaluate

$$I = \int_S (y^3 + 2x)\,dx + 3xy^2\,dy$$

where S is the straight line from $P = (0,0)$ to $Q = (1,2)$.

Sol^n (using FTC)

Consider

$$\omega = xy^3 + x^2$$

Then

$$d\omega = (y^3 + 2x)\,dx + 3xy^2\,dy$$

So

$$I = \int_S d\omega \overset{\text{FTC}}{=} \int_{\partial S} \omega = \omega|_Q - \omega|_P$$

$$= 9 - 0 = 9.$$

Alternative Solution

The points $(x = t,\ y = 2t)$ trace out the straight line segment from $P = (0,0)$ to $Q = (1,2)$, as t goes from $t = 0$ to $t = 1$.

$$x = t \qquad y = 2t$$

$$dx = dt \qquad dy = 2dt$$

$$I = \int_0^1 \left((2t)^3 + 2t\right) dt + 3t(2t)^2 2\,dt$$

$$= \int_0^1 32t^3 + 2t \ dt$$

$$= \frac{32t^4}{4} + t^2 \Big|_0^1$$

$$= 9$$

Problem Evaluate

$$I = \int_S (6xy^2 - y^3)\,dx + (6x^2y - 3xy^2)\,dy$$

where S is some curve from $P=(1,2)$ to $Q=(3,4)$.

Sol^n Try to find $w = F(x,y)$ such that

$$dw = F_x\,dx + F_y\,dy$$

$$= (6xy^2 - y^3)\,dx + (6x^2y - 3xy^2)\,dy$$

Well

$$F(x,y) = 3x^2y^2 - xy^3 + g(y)$$

$$F(x,y) = 3x^2y^2 - xy^3 + h(x)$$

we conclude that $g(y) = h(x) = c$ some constant.

$$\boxed{c = 0 \text{ say}}$$

$$I = \int_S d\omega$$

$$= \int_{\partial S} \omega = \int_{\partial S} \overbrace{3x^2y^2 - xy^3}^{F(x,y)}$$

$$= F(3,4) - F(1,2)$$

$$= (3 \cdot 9 \cdot 16 - 3 \cdot 64) - (3 \cdot 1 \cdot 4 - 1 \cdot 8)$$

$$= 236.$$

Continuity

A function $f(x,y)$ is continuous if a small change in input only ever produces a small change in output.

More formally, $f(x,y)$ is continuous at a point (x_0, y_0) if for any $\varepsilon > 0$ we can find a $\delta > 0$ such that $f(x,y)$ is defined and

$$|f(x,y) - f(x_0, y_0)| < \varepsilon$$

whenever $|x - x_0| < \delta$ and $|y - y_0| < \delta$.