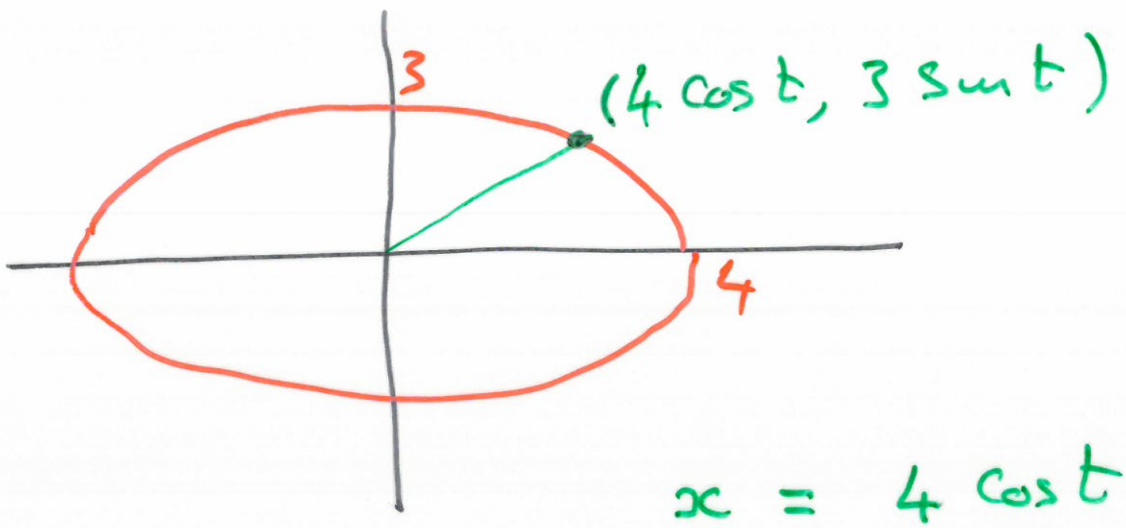
Sol^n

$$\text{work} = \int_S \omega$$

$$= \int_S (3x-4y)\,dx + (4x+2y)\,dy$$



$$x = 4\cos t$$
$$y = 3\sin t$$

$$\begin{cases} dx = -4\sin t \,.\, dt \\ dy = 3\cos t \,.\, dt \end{cases}$$

work =

$$\int_0^{2\pi} \Big(3(4\cos t) - 4(3\sin t)\Big)(-4\sin t)\,dt + \Big(4(4\cos t) + 2(3\sin t)\Big)3\cos t \,.\, dt$$

$= \cdots$

$$= \int_0^{2\pi} 48 - 30(\sin t)(\cos t)\, dt$$

$$= 48t - 15\sin^2 t \Big|_0^{2\pi}$$

$$= 96\pi.$$

Stokes' Formula

$$\boxed{\int_{\partial S} \omega = \int_S d\omega}$$

when $\omega = f(x_1, x_2, \cdots, x_n)$ is a 0-form, and S is a 1-dim oriented connected region:

- Left hand side makes sense to us;
- for the right hand side, we need to give a meaning to $d\omega$

We want to define the total derivative $d\omega$ of a 0-form ω. (Also called the exterior derivative.)

First:

Partial Derivatives

Given a 0-form

$$\omega = f(x, y, z)$$

we denote by

$$\frac{\partial f}{\partial x}$$

the 0-form obtained by regarding y and z as constants and differentiating with respect to x. We call $\frac{\partial f}{\partial x}$ the partial derivative of f with respect to x.

Example Consider

$$\omega = f(x, y, z) = \sqrt{1 - (x^2 + y^2 + z^2)}$$

defined on

$$S = \{(x,y,z) \in \mathbb{R}^2 : x^2+y^2+z^2 \leq 1\}.$$

Calculate $\dfrac{\partial f}{\partial x}$.

<u>Soln</u>

$$f(x,y,z) = (1-(x^2+y^2+z^2))^{\frac{1}{2}}$$

$$\frac{\partial f}{\partial x} = \frac{1}{2}\left(1-(x^2+y^2+z^2)\right)^{-\frac{1}{2}}(-2x)$$

$$= \frac{-x}{\sqrt{1-(x^2+y^2+z^2)}}.$$

similarly

$$\frac{\partial f}{\partial y} = \frac{-y}{\sqrt{1-(x^2+y^2+z^2)}}$$

$$\frac{\partial f}{\partial z} = \frac{-z}{\sqrt{1-(x^2+y^2+z^2)}}.$$

Notation

We often write

$$f_x$$

in place of

$$\frac{\partial f}{\partial x}.$$

Total derivatives

Given a 0-form

$$\omega = f(x, y, z)$$

then we define the 1-form

$$\boxed{d\omega = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz}$$

We call $d\omega$ the total derivative or exterior derivative of the 0-form ω.

<u>Example</u> Find the total derivative of the 0-form

$$\omega = \sqrt{1-(x^2+y^2+z^2)}$$

on $S = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \le 1\}$.

<u>Sol^n</u>

$$d\omega = \frac{-x}{\sqrt{1-(x^2+y^2+z^2)}}\, dx$$

$$- \frac{y}{\sqrt{1-(x^2+y^2+z^2)}}\, dy$$

$$- \frac{z}{\sqrt{1-(x^2+y^2+z^2)}}\, dz \, .$$