Continued from last Lecture :

$$\cos(x) = \cos\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) - \sin\left(\frac{x}{2}\right)\sin\left(\frac{x}{2}\right)$$

$$= \frac{1}{1+u^2} - \frac{u^2}{1+u^2}$$

$$= \frac{1-u^2}{1+u^2}$$

$$dx = 2\cos^2\left(\frac{x}{2}\right) du$$

$$dx = \frac{2}{1+u^2} du$$

So

$$w = \int \frac{1}{5+3\cos(x)} dx$$

$$= \int \left[\frac{1}{5+3\left(\frac{1-u^2}{1+u^2}\right)}\right] \frac{2}{1+u^2} du$$

$$= \ldots .$$

$$= \int \frac{1}{4+u^2} du$$

From log book

$$\omega = \frac{1}{2}\tan^{-1}\frac{u}{2} + C$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{1}{2}\tan\left(\frac{x}{2}\right)\right) + C.$$

This 0-form has total derivative

$$d\omega = \frac{1}{5+3\cos(x)}\,dx \quad .$$

Differential 0-forms on n-dimensional space

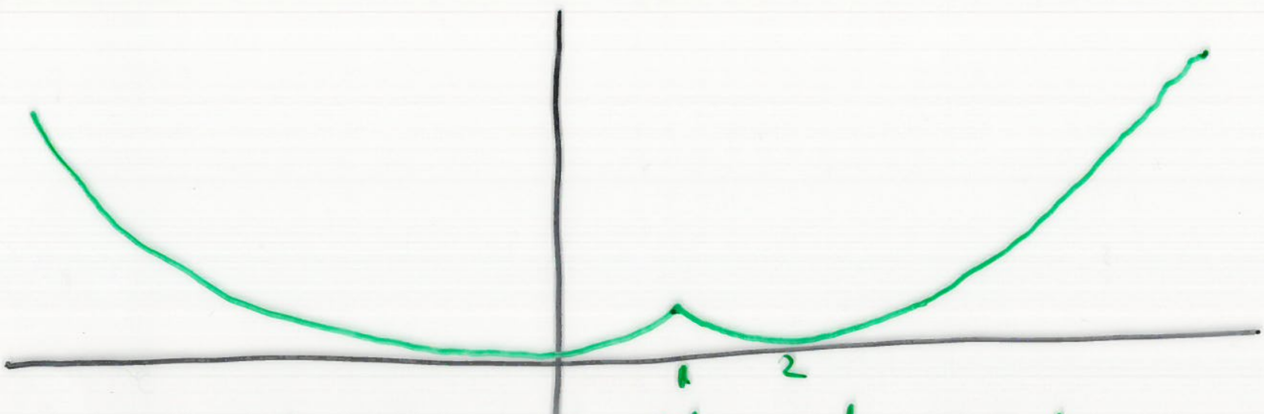
A differential 0-form on 2-dimensional space is a real valued function

$$\omega = f(x, y)$$

which is "differentiable". To explain this term, recall.

Informally: A function $f(x)$ is differentiable at a point if the curve $y = f(x)$ has a unique tangent line at x.

Example $y = \begin{cases} x^2 & x \leq 1 \\ (x-2)^2 & x > 1 \end{cases}$



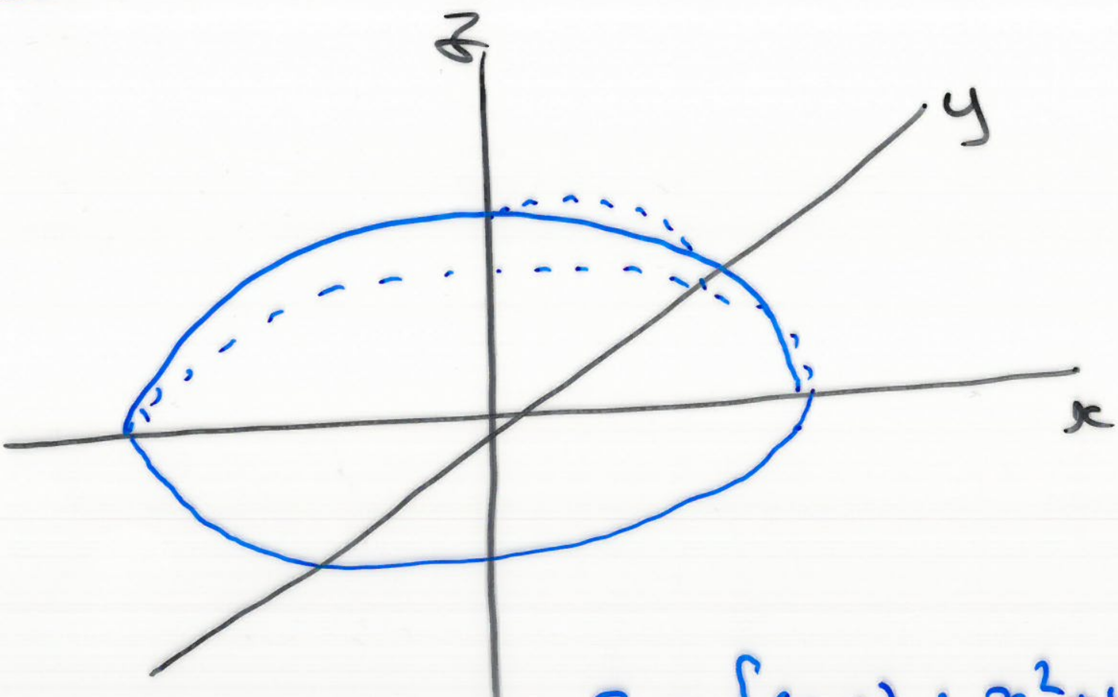
This is not differentiable at $x = 1$.

Informally A function $f(x,y)$ is differentiable at a point (x,y) if the surface

$$z = f(x,y)$$

has a well-defined (i.e. unique) tangent plane at (x,y).

Example $z = \sqrt{1-x^2-y^2}$ is defined for $x^2+y^2 \le 1$, and describes the surface



For any (x,y) in $S = \{(x,y) : x^2+y^2 \le 1\}$ the surface has a tangent plane.

So

$$w = \sqrt{1 - x^2 - y^2}$$

is a differential 0-form on S.

Let's skip the formal definition of differentiability.

Differential 1-forms on n-dimensional space

A differential 1-form on a 2-dimensional space is a function

$$w = A(x,y)\,h_1 + B(x,y)\,h_2$$

that inputs a vector (x,y) and a vector (h_1,h_2) and returns a real number. Here $A(x,y)$ and $B(x,y)$ must be differentiable.

Example Evaluate the 1-form

$$\omega = (x^2+y^2)\, h_1 + 2xy\, h_2$$

at $(x,y) = (2,4)$ and $(h_1,h_2) = (\frac{1}{4}, \frac{1}{4})$.

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Notation We usually denote

$$\omega = A(x,y)\, h_1 + B(x,y)\, h_2$$

by

$$\omega = A(x,y)\, dx + B(x,y)\, dy$$

Example Evaluate the 1-form

$$\omega = (x^2+y^2)\, dx + 2xy\, dy$$

at $(x,y) = (2,4)$, $(h_1,h_2) = (\frac{1}{4}, \frac{1}{4})$.

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