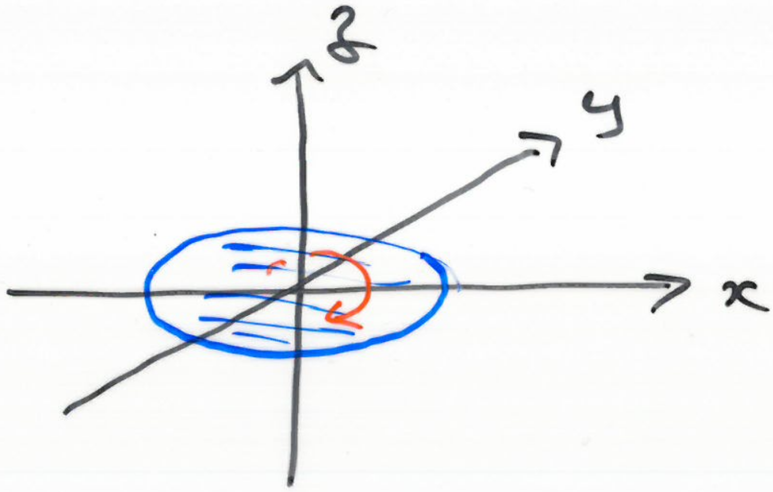
Example Evaluate

$$I = \int_S 3\,dx \wedge dy + 4\,dy \wedge dz$$

where S is the disk

$$S = \{ (x,y,z) \in \mathbb{R}^3 : z = 0,\ x^2+y^2 \leq 1 \}$$

with clockwise orientation.



Sol^n

From the definition of an integral we see that

$$I = 3 \times (-1) \text{area of disk} = -3\pi$$

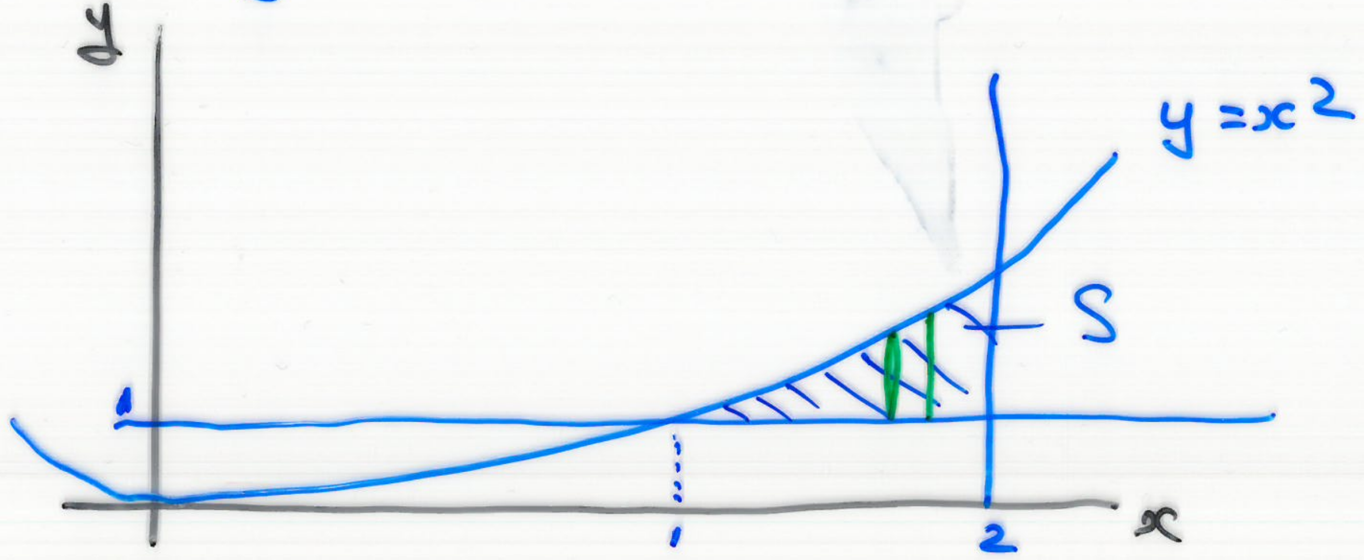
Example Let S be the region in the xy-plane bounded by $y = x^2$, $x = 2$, $y = 1$. Let S have an anti-clockwise orientation.

Evaluate

$$I = \int_S (x^2 + y^2 + z^2)\, dx \wedge dy$$

Sol^n On S we have $z = 0$ and thus

$$I = \int_S (x^2 + y^2)\, dx \wedge dy$$



Subdivide S into thin strips (3)
parallel to y-axis.

We can write

$$I = \int_{x=1}^{x=2} \left(\int_{y=1}^{y=x^2} (x^2+y^2)\, dy \right) dx$$

$$I = \int_{x=1}^{x=2} \left(x^2 y + \frac{y^3}{3} \Big|_{y=1}^{y=x^2} \right) dx$$

$$I = \int_1^2 \left(x^4 + \tfrac{1}{3}x^6 - x^2 - \tfrac{1}{3}\right) dx$$

$$= \frac{x^5}{5} + \frac{x^7}{21} - \frac{x^3}{3} - \frac{1}{3}x \Big|_{x=1}^{x=2}$$

$$= \frac{1006}{105}$$

Exercise Evaluate

$$I = \int_S (x+y+z)\, dx \wedge dy$$

where S is the oriented planar rectangle with vertices

$(0,0,1)$, $(0,1,1)$, $(1,1,1)$, $(1,0,1)$

in that order.

Differentiation of 1-forms

For a 1-form ω, and a 2-dimensional oriented region S, we'd like to define a 2-form

$$d\omega$$

such that

$$\boxed{\int_{\partial S} \omega = \int_S d\omega \qquad (*)}$$

The definition of $d\omega$ is determined by the desire to have equation $*$ hold.

Definition of the derivative of an n-form

For n-forms ω, ω'

$$\boxed{d(\omega+\omega') = d\omega + d\omega'}$$

For function $A, B, \dots$ in variables $x, y, z, \dots$

$$\boxed{dA = \frac{\partial A}{\partial x}dx + \frac{\partial A}{\partial y}dy + \frac{\partial A}{\partial z}dz + \cdots}$$

$$\boxed{\begin{aligned} &d(A\,dx + B\,dy + \cdots) = \\ &\quad dA \wedge dx + dB \wedge dy + \cdots \end{aligned}}$$

$$\boxed{dx \wedge dx = 0}$$

$$\boxed{dx \wedge dy = -dy \wedge dx}$$

Example Find $d\omega$

where

$$\omega = A\,dx + B\,dy$$

with A, B functions of x, y.

Soln

$$d\omega = d(A\,dx + B\,dy)$$

$$= d(A\,dx) + d(B\,dy)$$

$$= (dA \wedge dx) + (dB \wedge dy)$$

$$= \left(\frac{\partial A}{\partial x}dx + \frac{\partial A}{\partial y}dy\right) \wedge dx$$

$$+ \left(\frac{\partial B}{\partial x}dx + \frac{\partial B}{\partial y}dy\right) \wedge dy$$

$$= \cancel{\frac{\partial A}{\partial x} dx \wedge dx}^{\,=0} + \frac{\partial A}{\partial y} dy \wedge dx$$

$$+ \frac{\partial B}{\partial x} dx \wedge dy + \cancel{\frac{\partial B}{\partial y} dy \wedge dy}^{\,=0}$$

$$= \frac{\partial A}{\partial y} dy \wedge dx + \frac{\partial B}{\partial x} dx \wedge dy$$

$$= - \frac{\partial A}{\partial y} dx \wedge dy + \frac{\partial B}{\partial x} dx \wedge dy$$

$$= \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y} \right) dx \wedge dy .$$