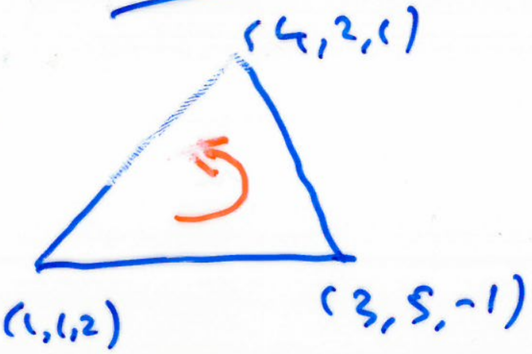
## Example

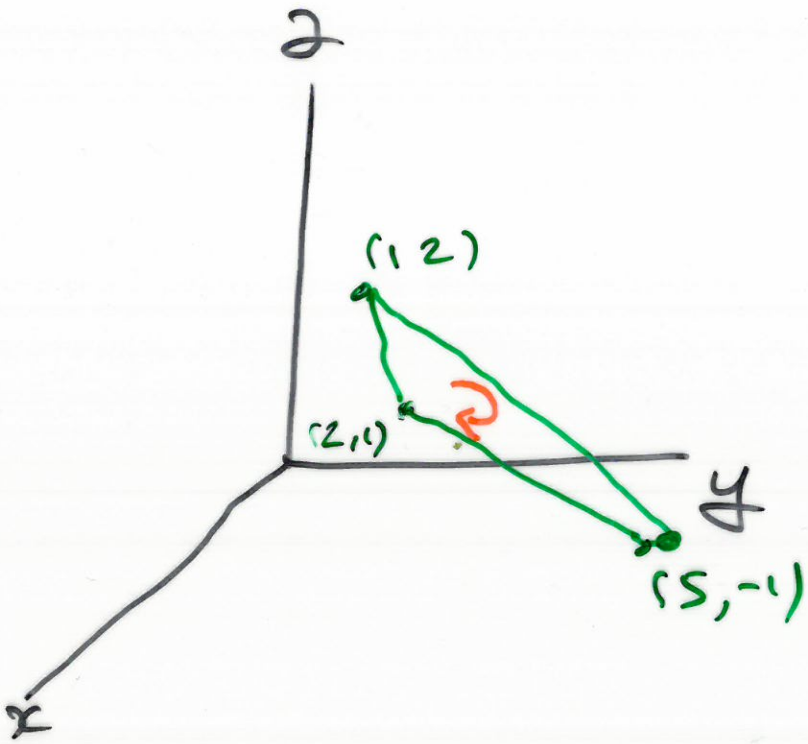
Evaluate

$$I = \int_S dy \wedge dz + dz \wedge dx$$

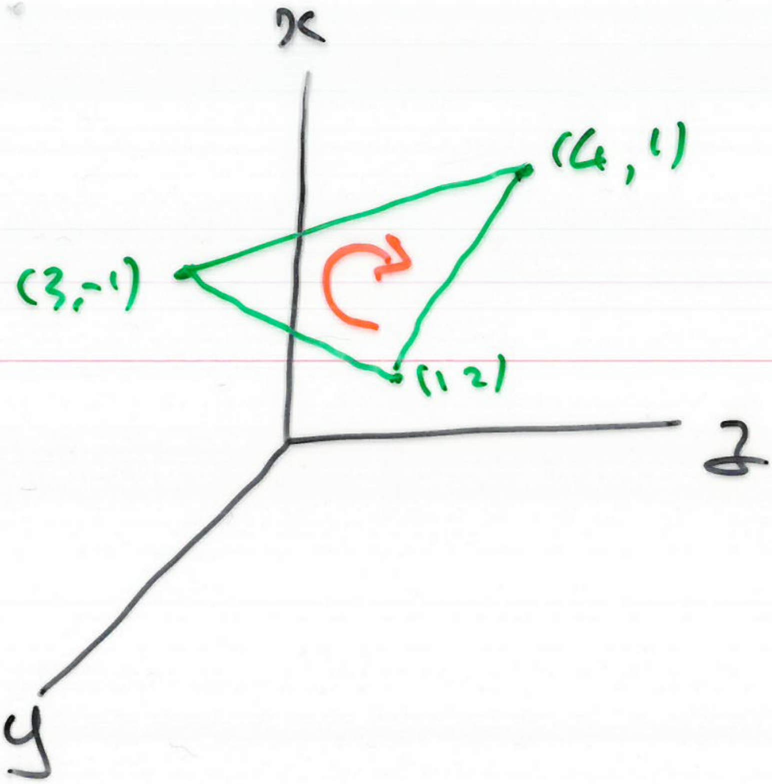
where $S$ is the oriented triangle with vertices $(1,1,2)$, $(3,5,-1)$, $(4,2,1)$ in that order.

Sol$^n$




$$\int_S dy \wedge dz = -\frac{1}{2}\begin{vmatrix} 1 & 4 \\ -1 & -3 \end{vmatrix} = -\frac{1}{2}$$



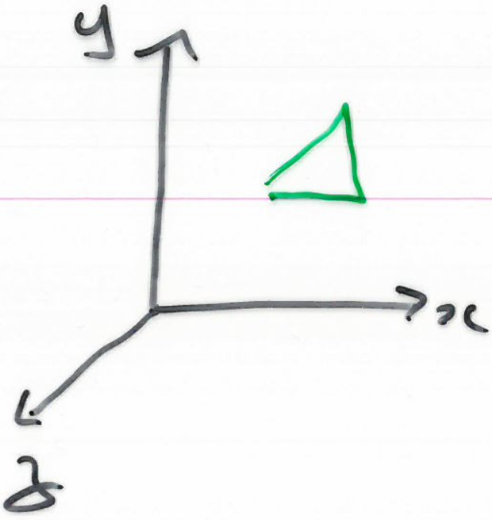

$$\int_S dz \wedge dx = -\frac{1}{2}\begin{vmatrix} 2 & 3 \\ -3 & -1 \end{vmatrix} = -\frac{7}{2}$$

So

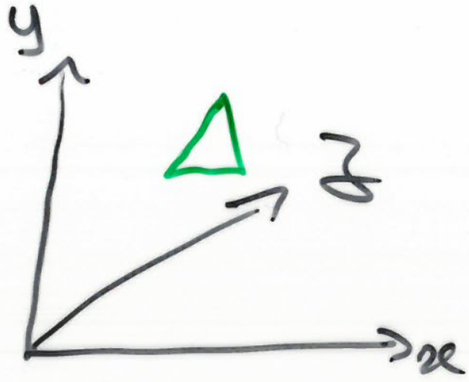
$$I = \int_S dy \wedge dz + dz \wedge dx = -\frac{1}{2} - \frac{7}{2} = -4$$

**Remark** $\int_S dx \wedge dy$ refers

to




$\int_S dy \wedge dx$ refers to




so

$$\boxed{\int_S dx \wedge dy = - \int_S dy \wedge dx}$$

we write

$$\boxed{dx \wedge dy = - dy \wedge dx}$$

Conventions

$dx \wedge dy$, $dy \wedge dz$, $dz \wedge dx$

correspond to "orientations" when we view things from the positive third axis.

So we write

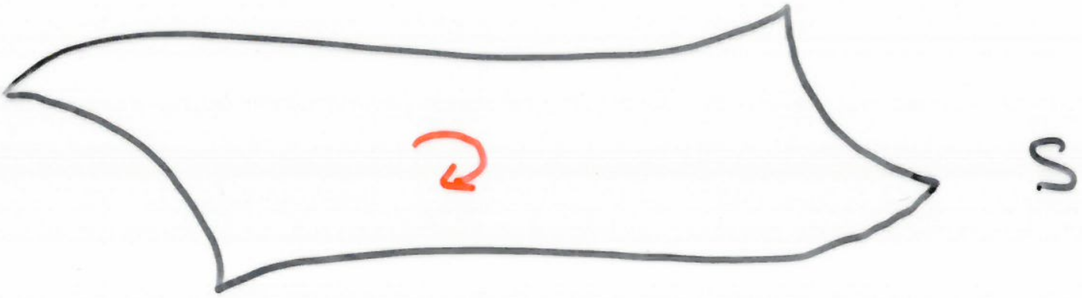
$$\int_S dy \wedge dz = - \int_S dz \wedge dy$$

and

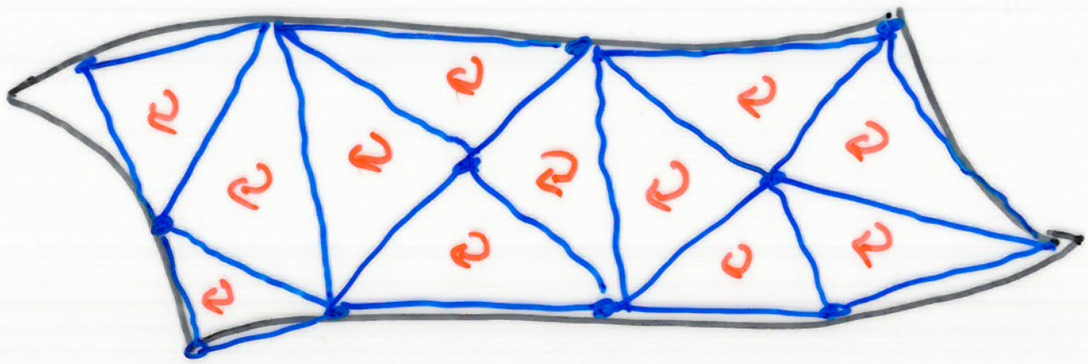
$$dy \wedge dz = - dz \wedge dy$$

# Integration of 2-forms

Let $S$ denote a 2-dimensional surface in $\mathbb{R}^3$ with some choice of orientation.



We can approximate $S$ by a union of oriented planar triangles.



$$P = T_1 \cup T_2 \cup \dots \cup T_k$$

union of $k$ oriented triangles.

Suppose we have a sequence $P_1, P_2, P_3 \dots$ of approximations to $S$, where:

1) the approximation $P_i$ gets better as $i \to \infty$.
2) the area of the largest triangle $P_i$ tends to $0$ as $i \to \infty$.

we define

$$\boxed{\begin{aligned}&\int_S A(x,y,z)\,dx\wedge dy + B(x,y,z)\,dy\wedge dz + C(x,y,z)\,dz\wedge dx \\ &= \lim_{i\to\infty} \sum_{T_j \text{ in } P_i} \int_{T_j} A(x_j,y_j,z_j)\,dx\wedge dy + B(x_j,y_j,z_j)\,dy\wedge dz + C(x_j,y_j,z_j)\,dz\wedge dx\end{aligned}}$$

where $(x_j, y_j, z_j)$ lies in $T_j$

**Example** Evaluate

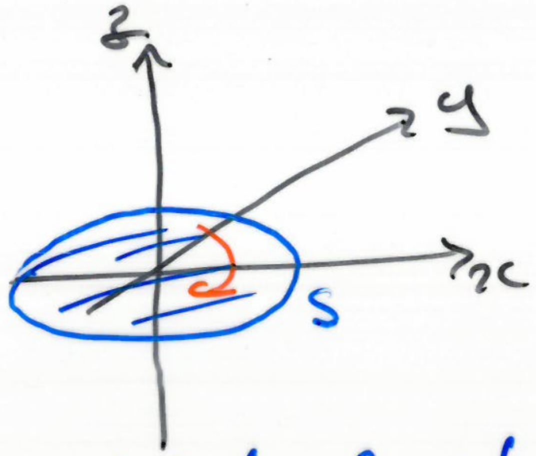
$$I = \int_S 3\, dx \wedge dy + 4\, dy \wedge dz$$

where $S$ is the disk

$$S = \{ (x,y,z) \in \mathbb{R}^3 : z = 0,\ x^2+y^2 \le 1 \}$$

with clockwise orientation.

**Sol^n**




From the definition of an integral we see that

$$I = \int_S 3\, dx \wedge dy = 3 \times (-1) \text{area of disk}$$

$$= -3\pi .$$