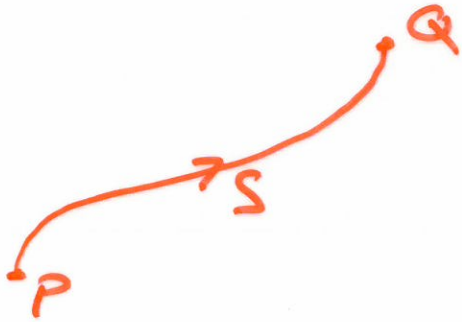
Proof of the fundamental theorem of Calculus

Want to prove

$$\int_S d\omega = \int_{\partial S} \omega$$



where ω is a 0-form.

For simplicity let's consider the case of $n=2$ variables.

Proof

Suppose $\omega = F(x,y)$

Suppose $x = g(t)$, $y = h(t)$ is a parametrization of S as t varies from t_0 to t_1.

$$\int_S dw = \int_S F_x(x,y)\,dx + F_y(x,y)\,dy$$

$$= \int_{t_0}^{t_1} F_x(g(t), h(t))\, g'(t)\, dt + F_y(g(t), h(t))\, h'(t)\, dt$$

$$= \int_{t_0}^{t_1} \Big(F_x(g(t), h(t))\, g'(t) + F_y(g(t), h(t))\, h'(t) \Big) dt$$

$$= \int_{t_0}^{t_1} \left(\frac{dF}{dt} \right) dt$$

$$\underset{\text{FTC in one variable}}{=} F(g(t_1), h(t_1)) - F(g(t_0), h(t_0))$$

$$= F(P) - F(Q)$$

$$= \int_{\partial S} w \qquad \text{QED.}$$

Summary of 1-forms and an introduction to 2-forms

- A 1-form is an expression such as

$$\omega = A\,dx + B\,dy$$

that can be integrated over oriented curves.

- A 2-form is an expression such as

$$\omega = A\,dx \wedge dy + B\,dy \wedge dz + C\,dz \wedge dx$$

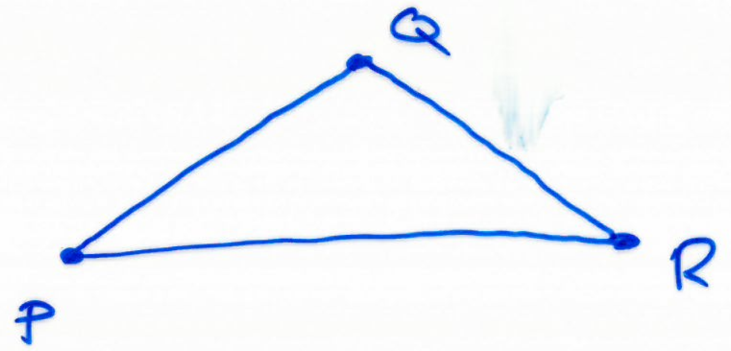
that can be "integrated" over 2-dimensional "oriented" regions

- integrals of 1-forms are just limits of sums of integrals of constant 1-forms over oriented straight line segments.

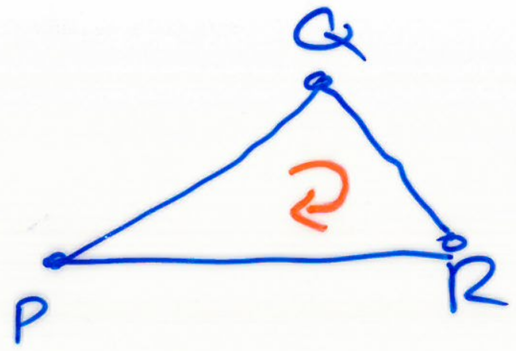
- integrals of 2-forms are just limits of sums of integrals of "constant 2-forms" over "oriented planar triangles".

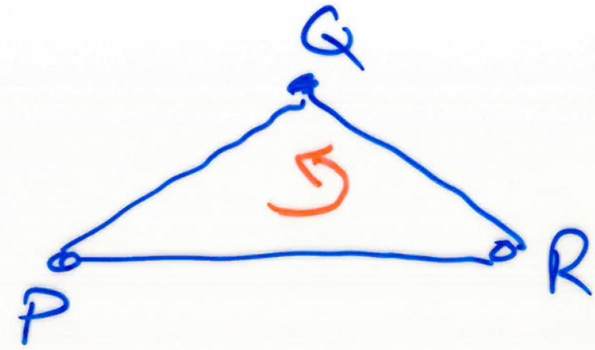
Oriented planar triangles

Three points in a plane determine a triangle



An orientation of a triangle is specified by an arrow





corresponding to one of two possible directions of rotation.

The positive side of the oriented triangle is the one

from which the arrow denotes anti-clockwise rotation.

An orientation is just an ordering of the vertices of the triangle. The ordering PRQ denotes the second triangle above. So too does RQP, and QPR.