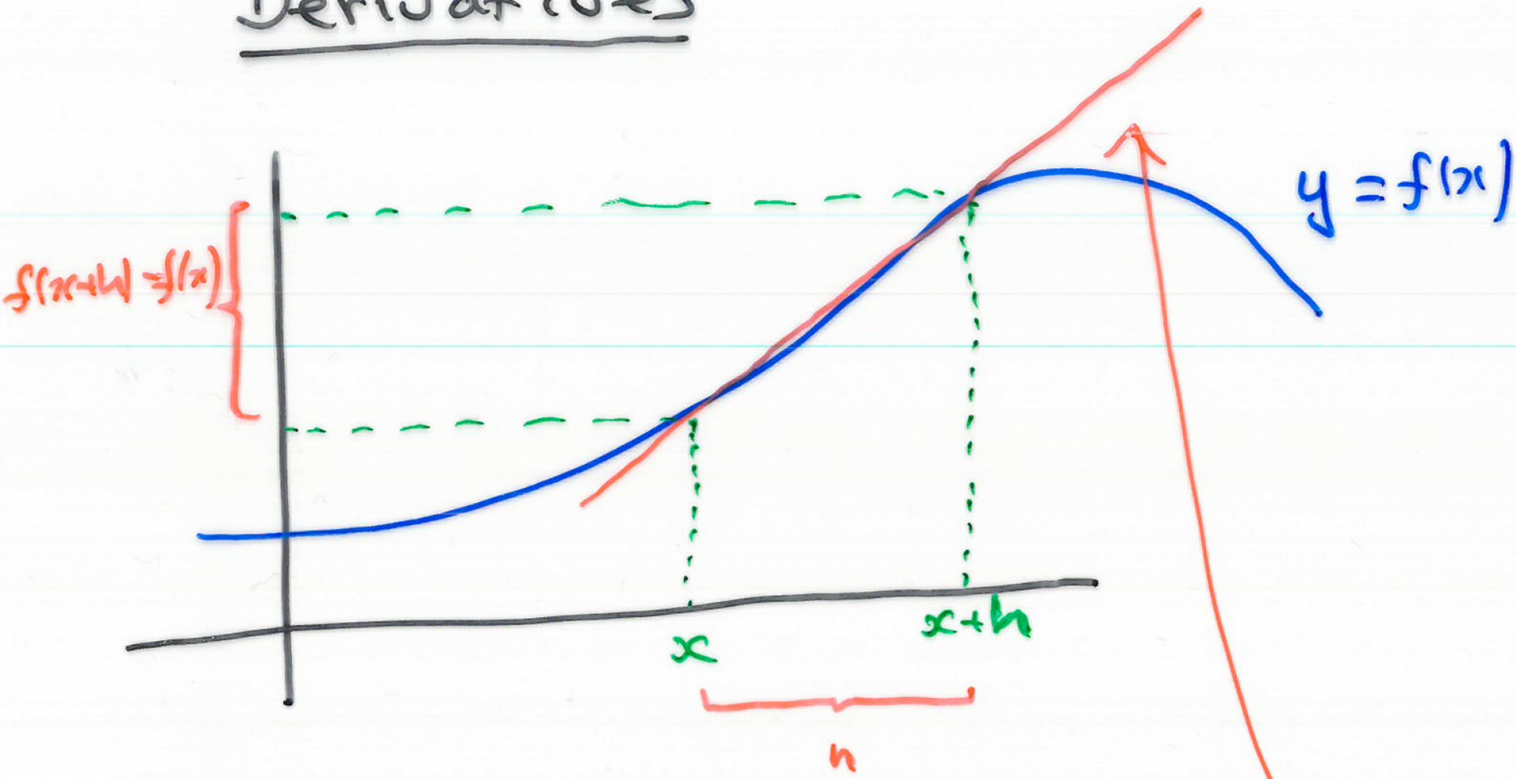
Derivatives



$$\frac{f(x+h)-f(x)}{h}$$

slope of this line

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

= slope of the tangent to the curve $y=f(x)$ at x.

Definition

Rules of differentiation

Rules of differentiation

- $$\boxed{\frac{d}{dx}\left(f(x) + g(x)\right) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)}$$

SUM RULE

This is obvious!

LHS =

$$\lim_{h\to 0} \frac{f(x+h) + g(x+h) - f(x) - g(x)}{h}$$

$$= \lim_{h\to 0}\left(\frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h}\right)$$

$$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$

$$= \frac{d}{dx}f(x) + \frac{d}{dx}g(x).$$

Example

$$\frac{d}{dx}\left(x^{\frac{3}{2}} + \sin(x)\right)$$

$$\underset{\text{Sum Rule}}{=} \frac{d}{dx} x^{\frac{3}{2}} + \frac{d}{dx}\sin(x)$$

$$= \frac{3}{2} x^{\frac{1}{2}} + \cos(x) .$$

- $$\boxed{\frac{d}{dx}\left(k f(x)\right) = k \frac{d}{dx} f(x)}$$

Scalar product rule
(k here is a constant)

Example

$$\frac{d}{dx}\left(3e^{x}\right) = 3 \frac{d}{dx} e^{x} = 3e^{x}$$

$$\frac{d}{dx} f(x)\, g(x)$$

$$= \left[\frac{d}{dx} f(x)\right] g(x) + f(x) \left[\frac{d}{dx} g(x)\right]$$

Product Rule

Example

$$\frac{d}{dx}\left(x^2 \sin(x)\right)$$

$$= \left[\frac{d}{dx} x^2\right] \sin(x) + x^2 \left[\frac{d}{dx} \sin(x)\right]$$

$$= 2x \sin(x) + x^2 \cos(x).$$

Example

$$y = (x^2+1)(x^3+2)$$

$$\frac{dy}{dx} = \left[\frac{d}{dx}(x^2+1)\right](x^3+2) + (x^2+1)\left[\frac{d}{dx}(x^3+2)\right]$$

$= 2x(x^3+2) + (x^2+1)3x^2$

$= \cancel{2x^4 + 2x^3 + 3x^2 +}$

$5x^4 + 3x^2 + 4x$

Chain Rule

Given functions

$f(x)$ and $g(x)$

we can consider the composite function

$y = g(f(x))$

- $\boxed{\dfrac{dy}{dx} = g'(f(x))\, f'(x)}$

Chain Rule

Example $y = \sin(x^2)$

$g(x) = \sin(x) \qquad f(x) = x^2$

$$\frac{dy}{dx} = \cos(x^2)\, 2x$$

$$= 2x\cos(x^2).$$

Example

$$y = (x^2 - x + 1)^7$$

$$\frac{dy}{dx} = 7(x^2 - x + 1)^6 . (2x - 1)$$

Example

$$y = \sqrt{x^2 + 1}$$

$$y = (x^2 + 1)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{1}{2}(x^2 + 1)^{-\frac{1}{2}}\, 2x$$

$$= \frac{x}{\sqrt{x^2 + 1}}$$

Quotient Rule

$$y = \frac{f(x)}{g(x)}$$

So

$$y = f(x)\,(g(x))^{-1}$$

$$\frac{dy}{dx} = f'(x)\,(g(x))^{-1}$$

$$- f(x)\,(g(x))^{-2}\,g'(x)$$

$$= \frac{f'(x)}{g(x)} - \frac{f(x)\,g'(x)}{g(x)^2}$$

$$= \frac{f'(x)\,g(x) - f(x)\,g'(x)}{g(x)^2}$$

Exercise: for $y = \dfrac{x^2+x-2}{x^3+6}$

find $\dfrac{dy}{dx}$