Limits at infinity

Defn

$$\boxed{\lim_{x \to \infty} f(x) = \lim_{x \to 0+} f\left(\frac{1}{x}\right)}$$

Defn

$$\boxed{\lim_{x \to -\infty} f(x) = \lim_{x \to 0-} f\left(\frac{1}{x}\right)}$$

Example Evaluate

$$l = \lim_{x \to \infty} \frac{1}{\sqrt{x^2+2x} - x}$$

Sol^n

$$l = \lim_{x \to \infty} \left(\frac{1}{\left(\sqrt{x^2+2x} - x\right)}\right) \frac{\left(\sqrt{x^2+2x} + x\right)}{\left(\sqrt{x^2+2x} + x\right)}$$

$$l = \lim_{x \to \infty} \frac{\sqrt{x^2+2x} + x}{x^2+2x - x^2}$$

$$l = \lim_{x \to \infty} \frac{\sqrt{x^2 + 2x} + x}{2x}$$

$$l = \lim_{x \to \infty} \left(\frac{\sqrt{x^2 + 2x}}{2x} + \frac{1}{2} \right)$$

$$l = \lim_{x \to \infty} \left(\frac{\sqrt{x^2 + 2x}}{\sqrt{4x^2}} + \frac{1}{2} \right)$$

$$l = \lim_{x \to \infty} \left(\sqrt{\frac{x^2 + 2x}{4x^2}} + \frac{1}{2} \right)$$

$$l = \lim_{x \to \infty} \left(\sqrt{\frac{1}{4} + \frac{1}{2x}} + \frac{1}{2} \right)$$

$$l = \sqrt{\frac{1}{4}} + \frac{1}{2}$$

$$l = \frac{1}{2} + \frac{1}{2}$$

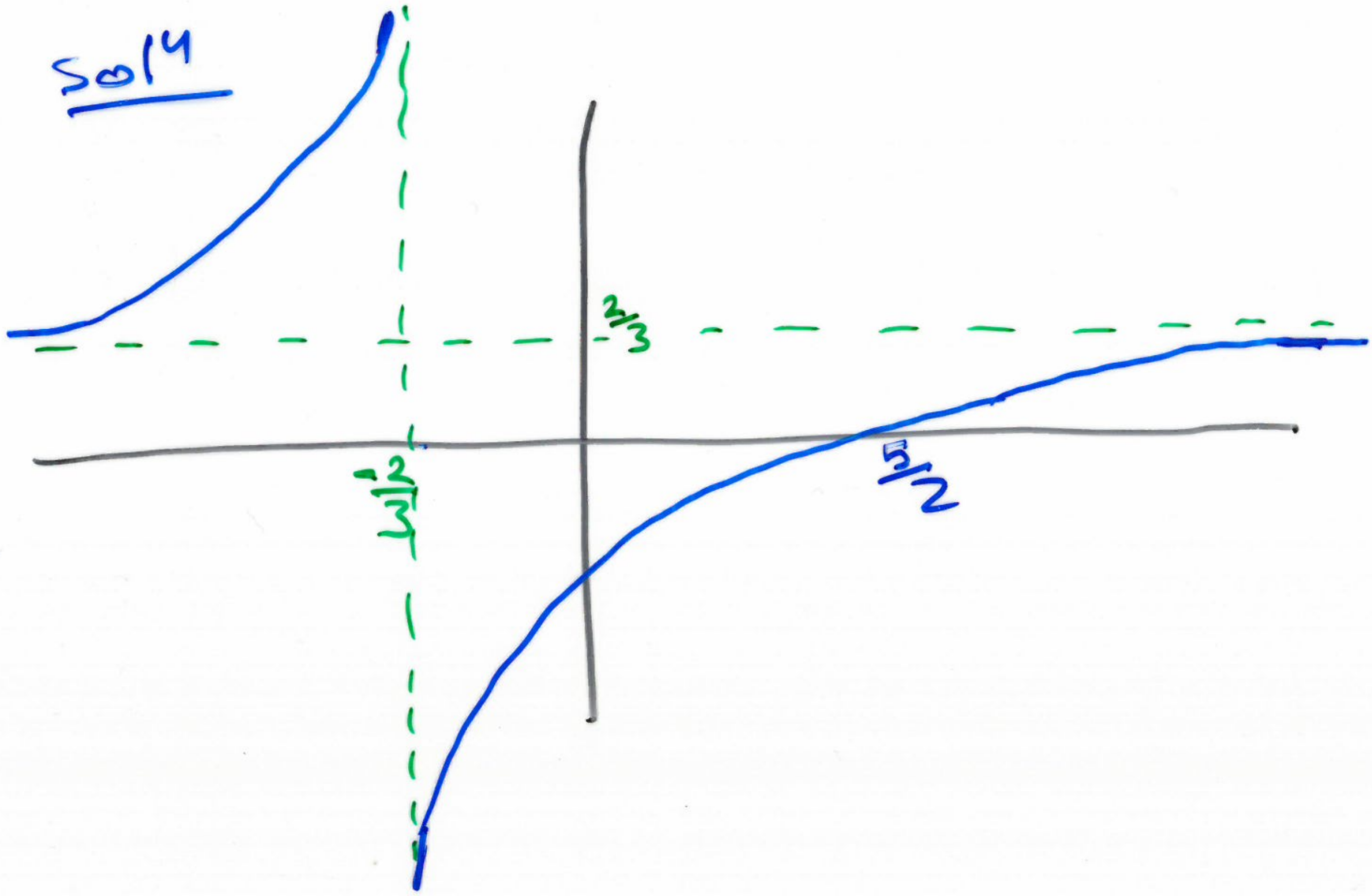
$$\underline{\underline{l = 1}}$$

Example what are the horizontal and vertical asymptotes of

$$y = \frac{2x-5}{3x+2} \quad ?$$

sketch the graph of y.

Sol^n



$$\lim_{x\to\infty} \frac{2x-5}{3x+2} = \frac{2}{3}$$

$$\lim_{x\to-\infty} \frac{2x-5}{3x+2} = \frac{2}{3}$$

So: limits at infinity correspond to horizontal asymptotes.

Topic II

Rates of change

Differentiation

Given a function $f(x)$ we define the derivative to be a function defined as

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

Example Find the derivative $f'(x)$ of the function $f(x) = x^2$.

Sol^n

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{2x\cancel{h} + h^{\cancel{2}}}{\cancel{h}}$$

$$= \lim_{h \to 0} 2x + h$$

$$= 2x.$$

So for $f(x) = x^2$ we have

$$f'(x) = 2x.$$

Derivatives of some basic functions

For $y = f(x)$ we often write

$$\frac{dy}{dx}$$

instead of

$$f'(x).$$

- $\frac{d}{dx} x^n = n x^{n-1}$ for any n.

- $\frac{d}{dx} \sin(x) = \cos(x)$

- $\frac{d}{dx} \cos(x) = -\sin(x)$

- $e = 2.71 \ldots$

$$\frac{d}{dx} e^x = e^x$$