Left-hand and right-hand limits

We write

$$\lim_{x \to a-} f(x) = \ell$$

to mean that $f(x)$ is close to ℓ for all x sufficiently close to a and strictly less than a.

Example

Let $f(x) = \begin{cases} x^2+1 & \text{if } x < 3 \\ x+8 & \text{if } x \geqslant 3 \end{cases}$

$$\lim_{x \to 3-} f(x) = 10$$

$f(3) = 11$

Analogously:

$$\lim_{x \to 3+} f(x) = 11$$

<u>Proposition</u>

$$\lim_{x \to a} f(x) = \ell$$

if and only if

$$\lim_{x \to a-} f(x) = \ell = \lim_{x \to a+} f(x).$$

If $\lim_{x \to a-} f(x) \neq \lim_{x \to a+} f(x)$

then $\lim_{x \to a} f(x)$ does not

exist.

Example

Let $f(x) = \begin{cases} x^2+1 & \text{if } x < 3 \\ x+c & \text{if } x \geq 3 \end{cases}$

where c is a constant.

For what value of c does

$$\lim_{x \to 3} f(x)$$

exist?

Sol

$c = 7$ because then

$$\lim_{x \to 3-} f(x) = 10$$

$$\lim_{x \to 3+} f(x) = 3+7 = 10.$$

Since $\lim_{x \to 3-} f(x) = 10 = \lim_{x \to 3+} f(x)$

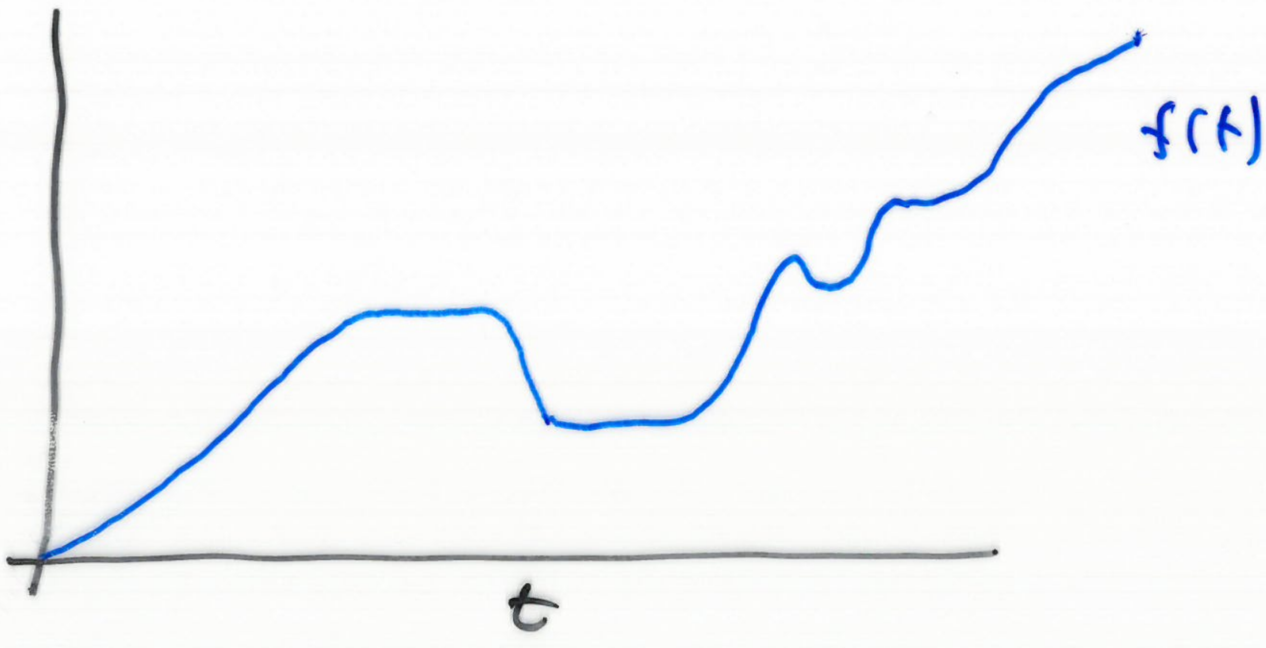
we have $\lim_{x \to 3} f(x) = 10.$

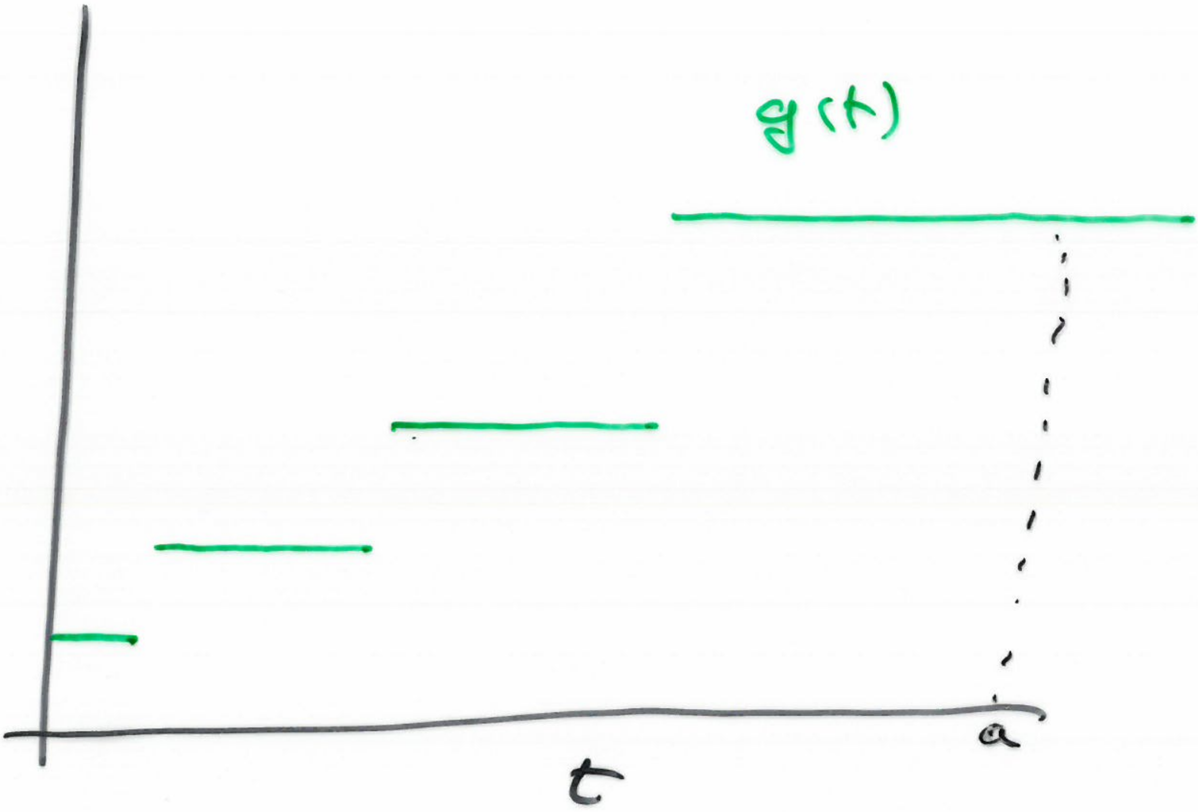
Continuity

I travel to Dublin airport.

$f(t)$ = distance from Galway t minutes after leaving.

$g(t)$ = price of parking my car t minutes after entering the car park.





Intuitively: continuous means there are no breaks in the graph.

Better definition:

A function $f(t)$ is continuous if a small change in the input can yield only a small change in the output.

In the above example $f(t)$ is continuous and $g(t)$ is not continuous.

Here is the best definition:

We say that $f(t)$ is continuous at a point $t=a$ if:

i) $f(t)$ is defined. (i.e. a is in the domain of f)

ii) $\lim_{t \to a} f(t)$ exists and

$$\lim_{t \to a} f(t) = f(a)$$

<u>Example</u> Determine the constant k such that

$$f(x) = \begin{cases} x^3 & \text{for } x \geq 2 \\ kx & \text{for } x < 2 \end{cases}$$

is continuous at all points.

<u>Soln</u>

The only issue is at $x = 2$.

We need

$$\lim_{x \to 2} f(x) = f(2) = 8$$

i.e. we need

$$\lim_{x \to 2-} f(x) = 8 = \lim_{x \to 2+} f(x)$$

$$\lim_{x \to 2-} f(x) = 2k$$

$$\lim_{x \to 2+} f(x) = 8$$

For continuity we need

$2k = 8$

or $k = 4$.