Composite function

Given two functions

$$f : D \to C, \quad g : C \to E$$

then we define their composite

$$g \circ f : D \longrightarrow E$$

by the formula

$$g \circ f(x) = g(f(x))$$

for $x \in D$.

Symmetries

- A function f is even if

$$f(x) = f(-x)$$

for all x in the domain of f.

- A function f is odd if

$$f(-x) = -f(x).$$

Example

- $\cos(x) = \cos(-x)$

So $\cos(x)$ is an even function.

- $\sin(-x) = -\sin(x)$

So $\sin(x)$ is odd.

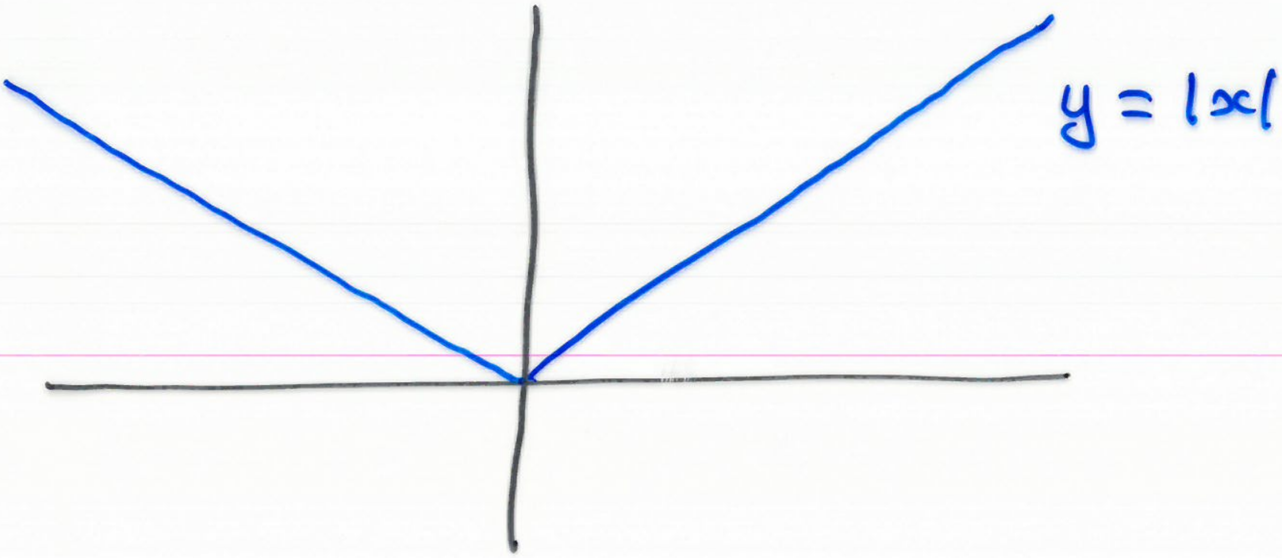
Example $|-3| = 3$

$|4| = 4$

$f(x) = |x|$ has domain $\mathbb{R}$.

This function is given by the formula

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$



This function is even since

$$|x| = |-x|$$

Proposition

Suppose

$$\lim_{x \to a} f(x) \quad \text{and} \quad \lim_{x \to a} g(x)$$

both exist. Then

i) $$\lim_{x \to a} \big(f(x) + g(x) \big) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x).$$

ii) $$\lim_{x \to a} k.f(x) = k \lim_{x \to a} f(x).$$

iii) $$\lim_{x \to a} \big(f(x).g(x) \big) = \left(\lim_{x \to a} f(x) \right) \left(\lim_{x \to a} g(x) \right)$$

iv) $$\lim_{x \to a} \left(\frac{f(x)}{g(x)} \right) = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$$

provided $\lim_{x \to a} g(x) \neq 0$.

Example

$$\lim_{x \to 2} \frac{x^2 + 4x + 5}{x^2 + 5}$$

(iv)

$$\frac{\lim_{x \to 2} (x^2 + 4x + 5)}{\lim_{x \to 2} (x^2 + 5)}$$

(i) & (ii)

$$\frac{\lim_{x \to 2} x^2 + 4 \lim_{x \to 2} x + \lim_{x \to 2} 5}{\lim_{x \to 2} x^2 + \lim_{x \to 2} 5}$$

$$= \frac{4 + 8 + 5}{4 + 5} = \frac{17}{9}$$

Sandwich Lemma

Suppose

$$f(x) \leq g(x) \leq h(x)$$

for all x near a (except possibly for $x=a$). Suppose also

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = l .$$

Then

$$\lim_{x \to a} g(x) = l .$$

Example Evaluate

$$\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) .$$

Sol^n $\quad g(x) = x^2 \sin\left(\frac{1}{x}\right)$

$f(x) = -x^2$

$h(x) = x^2$

for x near 0 and $x \neq 0$ we have

$$-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2$$

$$\lim_{x \to 0} -x^2 = 0 = \lim_{x \to 0} x^2$$

The Sandwich Lemma tells us

$$\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right) = 0.$$