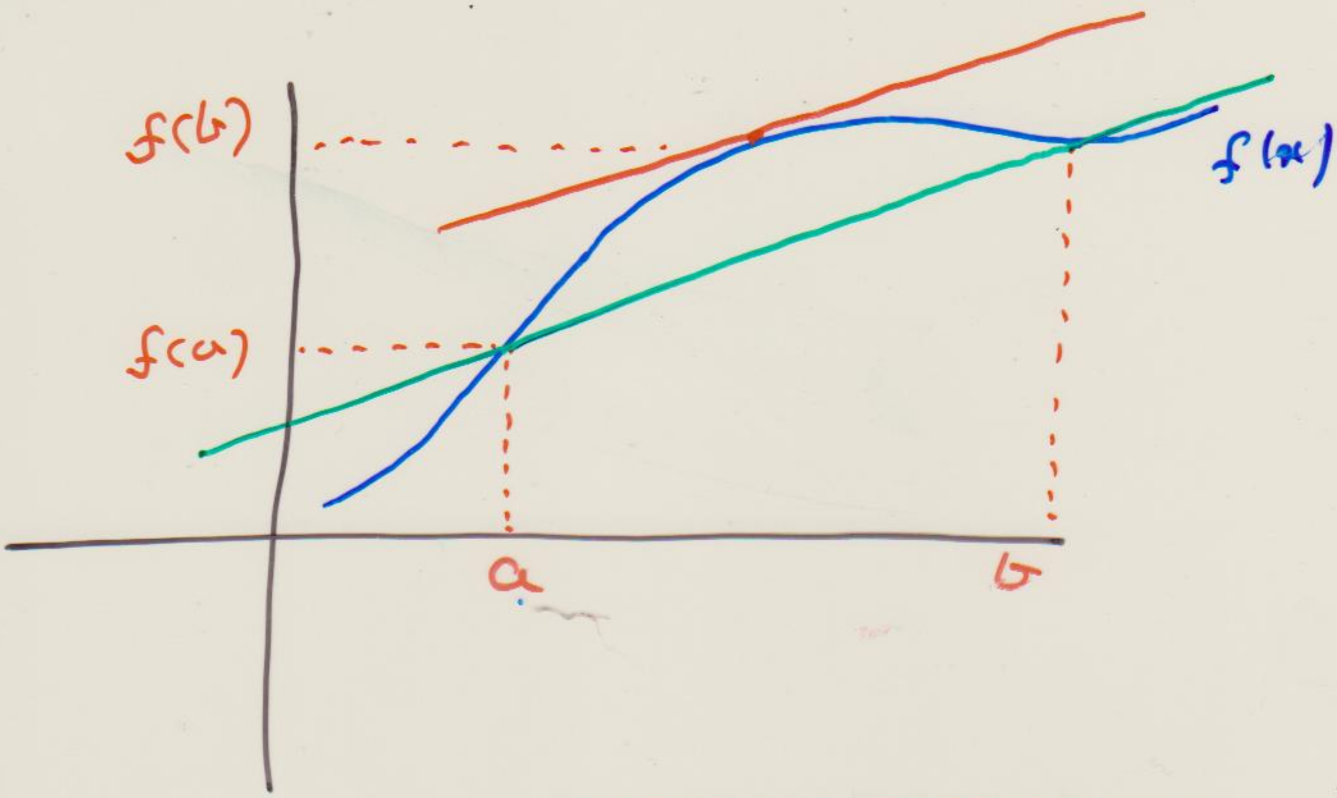
The Mean Value Theorem



Theorem Suppose that $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b) then there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Note: Rolle's Theorem is the special case where $f(a) = f(b)$.

Logarithms & Exponents

$4^2 = 16$

$4^3 = 64$

$4^{\frac{1}{2}} = \sqrt{4} = 2$

$4^{-2} = \frac{1}{4^2} = \frac{1}{16}$

$4^{\frac{3}{2}} = \left(4^{\frac{1}{2}}\right)^3 = 2^3 = 8$

$4^{\frac{5}{2}} = \left(4^{\frac{1}{2}}\right)^5 = 2^5 = 32$

$4^{\sqrt{2}} = ?$

$\log_4 16 = 2$

$\log_4 64 = 3$

$\log_4 2 = \frac{1}{2}$

$\log_4 \frac{1}{16} = -2$

$\log_4 8 = \frac{3}{2}$

$\log_4 32 = \frac{5}{2}$

We can make sense of 4^n when n is an integer ($0, 1, -1, 2, -2, \dots$) and when n is a fraction

$$4^{\frac{p}{q}} = \left(\sqrt[q]{4}\right)^p .$$

In calculus books fractions $\frac{p}{q}$ (where p, q are integers, $q \neq 0$) are called <u>rational numbers</u>.

<u>Theorem</u> $\sqrt{2}$ is not rational.

<u>Proof</u> Suppose $\sqrt{2}$ were rational.
So $\sqrt{2} = \frac{m}{n}$ for integers m, n with $n \neq 0$.
Without loss of generality we can assume $\gcd(m,n) = 1$.

$\sqrt{2} = \frac{m}{n}$ implies $2 = \frac{m^2}{n^2}$

and thus $m^2 = 2n^2$. $(*)$

So m^2 is even.
Consequently m must be even, let's say $m = 2M$.

From (*) we get

$$(2M)^2 = 2n^2$$

or

$$4M^2 = 2n^2$$

or

$$2M^2 = n^2.$$

Thus n^2 is even. Consequently n is even.

This is a contradiction: we can't have $\gcd(m,n) = 1$ and both m, n even.

We conclude that the original supposition is false.

Thus $\sqrt{2}$ is not rational.

$$\boxed{//}$$

Back to Calculus

$$\boxed{y = a^x \iff \log_a y = x}$$

$$\boxed{(a^m)^n = a^{mn}} \quad ①$$

$$\boxed{a^m a^n = a^{m+n}} \quad ②$$

Suppose
$u = a^m, \ v = a^n$. Then

$$\log_a uv = \log_a a^m a^n$$

$$\overset{②}{=} \log_a a^{m+n}$$

$$= m+n$$

$$= \log_a u + \log_a v$$

"Thus"

$$\boxed{\log_a uv = \log_a u + \log_a v}$$

(2')

Now let $u = a^m$. Then

$$\log_a u^n = \log_a (a^m)^n$$

$$\overset{1}{=} \log_a a^{mn}$$

$$= mn$$

$$= nm$$

$$= n \log_a u$$

"Thus"

$$\boxed{\log_a u^n = n \log_a u}$$

(1')